

Dantzig-Wolfe Reformulations for the Stable Set Problem

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Abstract. Dantzig-Wolfe reformulation of an integer program convexifies a subset of the constraints, which yields an extended formulation with a potentially stronger linear programming (LP) relaxation than the original formulation. This paper is part of an endeavor to understand the strength of such a reformulation in general.

We investigate the strength of Dantzig-Wolfe reformulations of the classical edge formulation for the maximum weighted stable set problem. Since every constraint in this model corresponds to an edge of the underlying graph, a Dantzig-Wolfe reformulation consists of choosing a subgraph and convexifying all constraints corresponding to edges of this subgraph. We characterize Dantzig-Wolfe reformulations not yielding a stronger LP relaxation (than the edge formulation) as reformulations where this subgraph is bipartite. Furthermore, we analyze the structure of facets of the stable set polytope and present a characterization of Dantzig-Wolfe reformulations with the strongest possible LP relaxation as reformulations where the chosen subgraph contains all odd holes (and 3-cliques).

To the best of our knowledge, these are the first non-trivial general results about the strength of relaxations obtained from decomposition methods, after Geoffrion's seminal 1974 paper about Lagrangian relaxation.

1 Introduction

The strength of relaxations and valid inequalities is a classical topic in polyhedral combinatorics. For Dantzig-Wolfe reformulations of integer programs “the” strength is non-obvious in general. This paper is a first step towards a clarification.

Let $n \in \mathbb{Z}_{>0}$ and let I be a finite index set. We are given the following integer program

$$(IP) \begin{aligned} \max & c^T x \\ \text{s. t.} & a_i^T x \leq b_i \quad \forall i \in I \\ & x \in \mathbb{Z}^n \cap [\ell, u] , \end{aligned}$$

where $\ell, u, c, a_i \in \mathbb{Q}^n$ and $b_i \in \mathbb{Q}$ for $i \in I$. The *integer hull* P_{IP} of (IP) is defined as

$$P_{IP} := \text{conv}\{x \in \mathbb{Z}^n \cap [\ell, u] : a_i^T x \leq b_i \quad \forall i \in I\} ,$$

whereas the *fractional polytope* P_{LP} contains all solutions that are feasible to the linear programming (LP) relaxation of (IP) , i.e.,

$$P_{LP} := \{x \in \mathbb{Q}^n \cap [\ell, u] : a_i^T x \leq b_i \ \forall i \in I\} .$$

Let $I' \subseteq I$ be a subset of the index set I and define the integer hull corresponding to the integer program that only consists of constraints with index in I' as

$$X(I') := \text{conv}\{x \in \mathbb{Z}^n \cap [\ell, u] : a_i^T x \leq b_i \ \forall i \in I'\} .$$

In Dantzig-Wolfe reformulation for integer programs every solution $x \in X(I')$ is reformulated as a convex combination of extreme points of $X(I')$. Thereby the variable vector x is replaced by new variables, one for each extreme point of $X(I')$, determining the coefficients in the convex combination. When we solve the LP relaxation of the new integer program, we implicitly optimize over the polytope

$$P_{DW}(I') := \{x \in \mathbb{Q}^n \cap [\ell, u] : a_i^T x \leq b_i \ \forall i \in I \setminus I', x \in X(I')\} ,$$

which corresponds to *convexifying* the constraints with index in I' [22]. We remark that the polytope $P_{DW}(I')$ is also obtained when the constraints with index in $I \setminus I'$ are dualized in Lagrangean relaxation [12] or when all valid inequalities for $X(I')$ are added to the LP relaxation of (IP) . Furthermore, note that optimizing over $P_{DW}(I')$ can be done in polynomial time if optimizing over $X(I')$ can be done in polynomial time.

We want to investigate the strength of Dantzig-Wolfe reformulations by analyzing the polytope $P_{DW}(I')$. We say that IP formulation A is *stronger* than IP formulation B if the set of LP-feasible solutions to A is strictly contained in the set of LP-feasible solutions to B . Obviously, the following inclusion relations hold:

$$P_{IP} \subseteq P_{DW}(I') \subseteq P_{LP} . \tag{1}$$

Thus, the Dantzig-Wolfe reformulation is potentially stronger than the original IP formulation (IP) . It is easy to see that the identities $P_{IP} = P_{DW}(I)$ and $P_{LP} = P_{DW}(\emptyset)$ hold, because this corresponds to convexifying all constraints or no constraint in the Dantzig-Wolfe reformulation. Moreover, it was proven in the context of Lagrangean relaxation by Geoffrion [12] that the relation $P_{DW}(I') \subsetneq P_{LP}$ holds only if

$$X(I') \subsetneq \{x \in \mathbb{Z}^n \cap [\ell, u] : a_i^T x \leq b_i \ \forall i \in I'\} .$$

Interestingly, this is already all we know about the strength of Dantzig-Wolfe reformulations of integer programs in general.

We start our endeavor of a better understanding of this strength with the investigation of Dantzig-Wolfe reformulations for the well-known stable set problem. On the one hand, the stable set problem was subject of a vast number of papers and the problem is well understood. On the other hand, the stable set problem plays an important role for general integer programs due its applications in the

conflict graph and the problem is strongly related to the set packing problem. We remark that in several applications set packing constraints occur in integer programs. Last, but not least, the stable set problem was used to understand other types of relaxations such as the Sherali-Adams, Lovász-Schrijver, and Lasserre relaxation [14], the corner relaxation [6, 9, 10], as well as some other families of relaxations [2].

The literature knows several Dantzig-Wolfe reformulations of the classical textbook model for the stable set problem. This model has a constraint for each edge (see below), so that we interchangeably use the notions of convexifying a subset of constraints and convexifying a subgraph. Warrier et al. [23] presented a branch-and-price approach for the stable set problem, where they apply Dantzig-Wolfe reformulation either by convexifying chordal induced subgraphs (such that optimizing over $P_{DW}(I')$ can be done in polynomial time) or by convexifying induced subgraphs using a heuristic partitioning of the node set of the graph. Sachdeva [20] extended the idea of Warrier et al. by using a partition of the edge set created from a partition of the node set to obtain a potentially stronger Dantzig-Wolfe reformulation. Ribeiro et al. [19] used a similar idea when they applied Lagrangean relaxation to stable set problems arising in map labeling problems. Campêlo and Corrêa [7] took a completely different approach and presented a Lagrangean relaxation based on representatives of stable sets. Another idea of Warrier et al. was picked up by Gabrel [11], who convexified perfect subgraphs such that optimizing over $P_{DW}(I')$ can again be done in polynomial time.

In contrast to all previous work, we investigate the strength of Dantzig-Wolfe reformulations for the stable set problem from a theoretical point of view. Furthermore, we do not limit the reformulation to convexifying a specific class of subgraphs. In particular we present a characterization of Dantzig-Wolfe reformulations for the stable set problem with $P_{LP} = P_{DW}(I')$ as well as a characterization of Dantzig-Wolfe reformulations with $P_{IP} = P_{DW}(I')$. To the best of our knowledge, after Geoffrion's 1974 paper [12], these are the first non-trivial general results about the strength of relaxations obtained from decomposition methods.

2 The Stable Set Problem

Let $G = (V, E)$ be a graph with $n := |V|$ nodes. A set of nodes $S \subseteq V$ is called *stable set* in G if no nodes of S are adjacent in G , i.e., $|e \cap S| \leq 1$ for all $e \in E$. Let $\mathcal{S}(G)$ be the set of all stable sets in G . A stable set S^* is called *maximum stable set* if $S^* \in \operatorname{argmax}\{|S| : S \in \mathcal{S}(G)\}$. The maximum cardinality $\alpha(G) := \max\{|S| : S \in \mathcal{S}(G)\}$ is called *stability number* of G . Given a weighting $w \in \mathbb{Z}_{\geq 0}^n$ on the nodes, a stable set S^* in G with weighting w is called *maximum weighted stable set* if $S^* \in \operatorname{argmax}\{\sum_{v \in S} w_v : S \in \mathcal{S}(G)\}$. The maximum weight $\alpha_w(G) := \max\{\sum_{v \in S} w_v : S \in \mathcal{S}(G)\}$ is called *weighted stability number* of G with weighting w . We will denote the weight of stable set S using weighting w with $w(S) := \sum_{v \in S} w_v$.

The classical way of formulating the maximum weighted stable set problem as an integer linear program is to introduce a binary variable $x_v \in \{0, 1\}$ for

each node $v \in V$ indicating whether node v is in the stable set ($x_v = 1$) or not ($x_v = 0$). Furthermore, we have a constraint for each edge $e \in E$ enforcing that at most one node in the stable set is incident to edge e . The objective function is to maximize the term $\sum_{v \in V} w_v \cdot x_v$ in order to maximize the weight of the stable set. This leads to the following integer linear programming formulation for the maximum weighted stable set problem, called *edge formulation*:

$$\begin{aligned} \max \quad & \sum_{v \in V} w_v \cdot x_v \\ \text{s. t.} \quad & x_u + x_v \leq 1 \quad \forall \{u, v\} \in E \\ & x \in \{0, 1\}^n . \end{aligned}$$

The *stable set polytope* $\text{STAB}(G)$ is defined as the convex hull of incidence vectors of stable sets in G , i.e.,

$$\text{STAB}(G) := \text{conv}\{x \in \{0, 1\}^n : x_u + x_v \leq 1 \ \forall \{u, v\} \in E\} .$$

The set of LP-feasible solutions for the edge formulation is denoted by $\text{ESTAB}(G)$ and is defined as

$$\text{ESTAB}(G) := \{x \in [0, 1]^n : x_u + x_v \leq 1 \ \forall \{u, v\} \in E\} .$$

We will refer to $\text{ESTAB}(G)$ as the *fractional stable set polytope*.

In Dantzig-Wolfe reformulation for integer programs a subset of the constraints is implicitly *convexified*. Let $E' \subseteq E$ be a subset of the edges of G and define $G' := (V, E')$. We will convexify all constraints corresponding to edges in E' :

$$\text{DW}(G, G') := \{x \in [0, 1]^n : x_u + x_v \leq 1 \ \forall \{u, v\} \in E \setminus E', \ x \in \text{STAB}(G')\} .$$

Specializing (1), the previously defined polytopes relate as

$$\text{STAB}(G) \subseteq \text{DW}(G, G') \subseteq \text{ESTAB}(G) .$$

We want to investigate the strength of Dantzig-Wolfe reformulations by investigating the polytope $\text{DW}(G, G')$. Especially, we are interested in conditions for G' such that either $\text{STAB}(G) = \text{DW}(G, G')$ or $\text{DW}(G, G') = \text{ESTAB}(G)$ holds. Obviously, the identity $\text{STAB}(G) = \text{DW}(G, G)$ holds, because every constraint is convexified. In this case the reformulation is *strongest possible*. If we choose $E' = \emptyset$, the identity $\text{ESTAB}(G) = \text{DW}(G, G')$ holds and the reformulation is *weakest possible*. In this paper we present a characterization of Dantzig-Wolfe reformulations that are weakest possible as well as a characterizations of strongest possible Dantzig-Wolfe reformulations.

3 Weakest Possible Dantzig-Wolfe Reformulations

Nemhauser and Trotter [17] characterized graphs G such that the stable set polytope $\text{STAB}(G)$ is equal to the fractional stable set polytope $\text{ESTAB}(G)$.

Proposition 1 ([17]). *Let $G = (V, E)$ be a graph. Then $\text{ESTAB}(G) = \text{STAB}(G)$ holds if and only if G is bipartite.*

Thus, if G is bipartite, all three polytopes $\text{ESTAB}(G)$, $\text{DW}(G, G')$, and $\text{STAB}(G)$ coincide, no matter how we choose the graph G' . Note that the graph G' is always bipartite in this case, because G is bipartite.

A graph G is bipartite if and only if G does not contain any odd cycle. Let $C = (V_C, E_C)$ be an odd cycle in G with $|V_C| = 2k + 1$ for some $k \in \mathbb{Z}_{>0}$. The following *odd cycle inequality* [18] is valid for the stable set polytopes $\text{STAB}(G[V_C])$ and $\text{STAB}(G)$:

$$\sum_{v \in V_C} x_v \leq k .$$

We will use Prop. 1 and odd cycle inequalities in order to prove the following characterization of weakest possible Dantzig-Wolfe reformulations:

Theorem 1. *Let $G = (V, E)$ be a graph, let $E' \subseteq E$ be a subset of the edges, and define $G' := (V, E')$. Then $\text{DW}(G, G') = \text{ESTAB}(G)$ if and only if G' is bipartite.*

Proof. Consider the definitions of the polytopes:

$$\begin{aligned} \text{DW}(G, G') &= \{x \in [0, 1]^n : x_u + x_v \leq 1 \ \forall \{u, v\} \in E \setminus E', \ x \in \text{STAB}(G')\} , \\ \text{ESTAB}(G) &= \{x \in [0, 1]^n : x_u + x_v \leq 1 \ \forall \{u, v\} \in E\} \\ &= \{x \in [0, 1]^n : x_u + x_v \leq 1 \ \forall \{u, v\} \in E \setminus E', \ x \in \text{ESTAB}(G')\} . \end{aligned}$$

It is easy to see that $\text{STAB}(G') = \text{ESTAB}(G')$ implies $\text{DW}(G, G') = \text{ESTAB}(G)$.

Now suppose $\text{DW}(G, G') = \text{ESTAB}(G)$ holds and assume that $\text{STAB}(G') \neq \text{ESTAB}(G')$. Then there exists an odd cycle $C = (V_C, E_C)$ in G' . Let \bar{x} be the solution with $\bar{x}_v = \frac{1}{2}$ for $v \in V_C$ and $\bar{x}_v = 0$ otherwise. The solution \bar{x} is obviously in $\text{ESTAB}(G)$, but \bar{x} is not in $\text{STAB}(G')$, because it does not satisfy the odd cycle inequality corresponding to the odd cycle C . Since $\text{STAB}(G') \subseteq \text{DW}(G, G')$, this implies $\bar{x} \notin \text{DW}(G, G')$ and therefore $\text{DW}(G, G') \neq \text{ESTAB}(G)$, which is a contradiction.

Thus, the equation $\text{DW}(G, G') = \text{ESTAB}(G)$ holds if and only if $\text{STAB}(G') = \text{ESTAB}(G')$. Together with Prop. 1 this proves the theorem. \square

Due to Thm. 1, we obtain a stronger relaxation using Dantzig-Wolfe reformulation if and only if G' is not bipartite. Hence, for this to happen, it is necessary that the graph G' contains some odd cycle.

4 Strongest Possible Dantzig-Wolfe Reformulations

4.1 Necessary Condition

We have seen that bipartite graphs and odd cycles are important when we consider the strength of Dantzig-Wolfe reformulations of the edge formulation.

In this section we derive a necessary condition for the graph G' such that the Dantzig-Wolfe reformulation is strongest possible. In order to do this, we consider a particular type of odd cycles.

A subgraph $H = (V_H, E_H)$ of G is called *odd hole* if H is an induced cycle with an odd number of nodes. An edge e is called *chord* of a cycle $C = (V_C, E_C)$ in G if $e \notin E_C$, but $e \in E(G[V_C])$ with $E(G[V_E])$ being the edge set of the subgraph $G[V_C]$ of G induced by V_C . Hence, an odd hole H is a chordless odd cycle, i.e., an odd cycle without chords. Note that in contrast to other definitions in the literature, a cycle on three nodes is also considered an odd hole in this paper. Using this definition, it is obvious that a graph G is bipartite if and only if it does not contain any odd hole. Note that a graph contains an odd hole if and only if it contains an odd cycle.

Odd holes play an important role when we consider odd cycle inequalities. An odd cycle inequality corresponding to cycle C defines a facet of $\text{STAB}(G[V_C])$ if the cycle C is an odd hole in G [18].

In the following theorem we state that G' has to contain *all* odd holes if we want to obtain the strongest possible Dantzig-Wolfe reformulation.

Theorem 2. *Let $G = (V, E)$ be a graph with $n := |V|$, let $E' \subseteq E$ be a subset of the edges, and define $G' := (V, E')$. If $\text{DW}(G, G') = \text{STAB}(G)$ holds, then G' contains all odd holes of G .*

Proof. Suppose that $\text{DW}(G, G') = \text{STAB}(G)$ holds and assume there exists an odd hole $H = (V_H, E_H)$ that is not contained in G' , i.e., $E_H \not\subseteq E'$. Hence, there exists an edge $e \in E_H$ with $e \notin E'$. Let $V_H = \{v_1, v_2, \dots, v_{2k+1}\}$ and

$$E_H = \{\{v_i, v_{i+1}\} : i = 1, \dots, 2k\} \cup \{\{v_1, v_{2k+1}\}\}$$

for some $k \in \mathbb{Z}_{>0}$. Furthermore, let $e = \{v_1, v_{2k+1}\}$.

The solution \bar{x} with $\bar{x}_v = \frac{1}{2}$ for $v \in V_H$ and $\bar{x}_v = 0$ otherwise is obviously not in $\text{STAB}(G)$, because the odd cycle inequality $\sum_{v \in V_H} x_v \leq k$ is not satisfied. The solution \bar{x} is a convex combination of incidence vectors x^{even} and x^{odd} of the stable sets $S_{\text{even}} := \{v_{2l} : l = 1, \dots, k\}$ and $S_{\text{odd}} := \{v_{2l+1} : l = 0, \dots, k\}$ in G' , respectively, using coefficients $\frac{1}{2}$ for both incidence vectors, i.e., $\bar{x} = \frac{1}{2}x^{\text{even}} + \frac{1}{2}x^{\text{odd}}$. Thus, $\bar{x} \in \text{STAB}(G')$ holds. Furthermore, the edge inequalities $x_u + x_v \leq 1 \ \forall \{u, v\} \in E \setminus E'$ are satisfied, which implies that $\bar{x} \in \text{DW}(G, G')$ holds. This contradicts the assumption that $\text{DW}(G, G') = \text{STAB}(G)$. \square

Note that Thm. 2 is not true if we replace odd holes by odd cycles. The graph $G = (V, E)$ with $V = \{v_1, \dots, v_5\}$ and $E = \{\{v_i, v_{i+1}\} : 1 \leq i \leq 4\} \cup \{\{v_5, v_1\}, \{v_2, v_5\}\}$ gives a counter example. If we choose the edge set of the graph G' as $E' := \{\{v_1, v_2\}, \{v_2, v_5\}, \{v_5, v_1\}\}$, we already obtain $\text{DW}(G, G') = \text{STAB}(G)$, although the odd cycle $C = (V_C, E_C)$ with $V_C = V$ and $E_C = E \setminus \{\{v_2, v_5\}\}$ is not covered by G' .

4.2 Sufficient Condition

In the following we will prove that it is sufficient to cover all odd holes G with the graph G' in order that the Dantzig-Wolfe reformulation is strongest possible.

But before we prove this, we analyze the structure of facets of the stable set polytope.

The *trivial inequalities* $x_v \geq 0$ for $v \in V$ are facets of $\text{STAB}(G)$. All other inequalities are called *non-trivial* facets. The edge inequalities $x_u + x_v \leq 1$ for $\{u, v\} \in E$ are facets (called *edge-facets*) if and only if $\{u, v\}$ is a maximal clique [18]. Let $\sum_{v \in V} \pi_v x_v \leq \pi_0$ be a non-trivial, non-edge facet of $\text{STAB}(G)$. Then $\pi_v \geq 0$ for all $v \in V$, $\pi_0 > 0$, and $\pi_0 = \alpha_\pi(G)$ holds [15, 16, 21]. Let $V_0 := \{v \in V : \pi_v > 0\}$ be the set of nodes corresponding non-zero entries of π and let $G_0 := G[V_0] = (V_0, E_0)$ be the subgraph of G induced by V_0 . Obviously, the inequality $\sum_{v \in V_0} \pi_v x_v \leq \pi_0$ defines a facet of $\text{STAB}(G_0)$ and the identity $\pi_0 = \alpha_\pi(G_0)$ holds. A graph G_0 with weights $\pi_v > 0$ for $v \in V_0$ will be called *facet-graph* [15, 16] if $\sum_{v \in V_0} \pi_v x_v \leq \pi_0$ defines a facet of $\text{STAB}(G_0)$.

The following useful lemma gives us an idea of the special structure of facet-graphs.

Lemma 1 ([21]). *Let $G_0 = (V_0, E_0)$ with weights $\pi_v > 0$ for all $v \in V_0$ be a connected facet-graph with $|V_0| \geq 3$ and let $\{u, v\} \in E_0$ be an edge. Then there exists a maximum weighted stable set in G_0 that contains neither u nor v .*

The following definition of [15, 16, 21] will be important for analyzing the structure of facet-graphs. An edge $e \in E$ in a graph G with weighting π is called *critical* if the maximum weight $\alpha_\pi(G - \{e\})$ of a stable set in $G - \{e\}$ with weighting π obtained from G by deleting e is smaller than the maximum weight $\alpha_\pi(G)$ of a stable set in G with weighting π . A graph G with weighting π is called α_π -*critical* if every edge $e \in E$ is critical.

The class of α_π -critical graphs contains the well-known α -critical graphs [8], which are α_π -critical graphs with weights $\pi_v = 1$ for all $v \in V$. The following result is related to similar results for α -critical graphs [1, 3, 4] and the proof idea was already used by Andrásfai [1].

Lemma 2. *Let $G_0 = (V_0, E_0)$ with weights $\pi_v > 0$ for all $v \in V_0$ be a connected facet-graph with $|V_0| \geq 3$ and let $e \in E_0$ be a critical edge. Then there exists an odd hole $H = (V_H, E_H)$ in G_0 containing the edge e , i.e., $e \in E_H$.*

Proof. Let $e = \{u, v\} \in E_0$ be a critical edge. Lem. 1 implies that there exists a maximum weighted stable set S in G_0 with $u, v \notin S$. Let S^+ be a maximum weighted stable set in $G_0^+ := G_0 - \{e\}$. Obviously, $\pi(S^+) > \pi(S)$ and $u, v \in S^+$ (otherwise S^+ would be an stable set in G_0 with larger weight than S). Consider the induced bipartite subgraph $G_{bip} := G_0[S^+ \setminus S \cup S \setminus S^+]$ with bipartition $(S^+ \setminus S, S \setminus S^+)$.

Assume that u and v are in different connected components of G_{bip} . Since

$$\pi(S^+ \setminus S) = \pi(S^+) - \pi(S^+ \cap S) > \pi(S) - \pi(S^+ \cap S) = \pi(S \setminus S^+)$$

holds, there exists a connected component K of G_{bip} with

$$\pi(S^+ \setminus S \cap K) > \pi(S \setminus S^+ \cap K) .$$

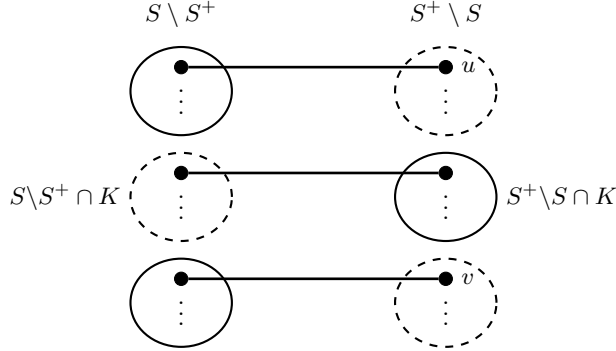


Fig. 1. A sketch of the connected components of the graph G_{bip} and the stable set S^{++} depicted with solid border (under the assumption that u and v are in different connected components of G_{bip}).

Since

$$N(S^+ \setminus S \cap K) \cap S = S \setminus S^+ \cap K$$

and $|\{u, v\} \cap K| \leq 1$, the set

$$S^{++} := (S \cap S^+) \cup (S \setminus K) \cup (S^+ \setminus S \cap K)$$

is a stable set in G_0 as sketched in Fig. 1.

The following chain of inequalities holds

$$\pi(S^{++}) = \pi(S^{++} \cap S) + \pi(S^+ \setminus S \cap K) > \pi(S^{++} \cap S) + \pi(S \setminus S^+ \cap K) = \pi(S) .$$

This contradicts the condition that S is a maximum weighted stable set in G_0 . Hence, u and v are in the same connected component of G_{bip} .

Let $P = (u, w_1, \dots, w_l, v)$ be some u - v -path of shortest length in G_{bip} . Note that P is an induced path in G_{bip} . Since u and v are contained in the same partite set, the path has even length, i.e., $l + 1 = 2k$ for some $k \in \mathbb{Z}_{>0}$. The odd cycle $H = (V_H, E_H)$ of G_0 with $V_H := \{u, v\} \cup \{w_i : i = 1, \dots, l\}$ and $E_H := \{\{u, w_1\}, \{w_l, v\}, \{u, v\}\} \cup \{\{w_i, w_{i+1}\} : i = 1, \dots, l - 1\}$ is chordless and contains the edge e , which concludes the proof. \square

Since not every edge of a facet-graph is critical, we will also use a more general definition by considering subset of edges. A subset $C \subseteq E$ of the edges is called *critical* if $\alpha_\pi(G - C) > \alpha_\pi(G)$. The critical set C of edges is *minimal* if $\alpha_\pi(G - (C \setminus \{e\})) = \alpha_\pi(G)$ holds for all $e \in C$. Note that a set $C = \{e\}$ containing exactly one edge e is critical if and only if the edge e is critical.

The following lemma generalizes the result of Lem. 2 from critical edges to critical sets of edges.

Lemma 3. *Let $G_0 = (V_0, E_0)$ with weights $\pi_v > 0$ for all $v \in V_0$ be a connected facet-graph with $|V_0| \geq 3$ and let $C \subseteq E_0$ be a minimal critical set of edges. Then*

there exists an edge $e \in C$ that is contained in an odd hole $H = (V_H, E_H)$ in G_0 , i.e., $e \in E_H$.

Proof. Since C is a critical set of edges, each edge $e \in C$ is critical in $G_0 - (C \setminus \{e\})$. Let $C = \{e_0, \dots, e_{|C|-1}\}$ and $C_k := \{e_i : k \leq i \leq |C| - 1\}$ for all $k \in \mathbb{Z}$ with $1 \leq k \leq |C| - 1$. We will proof by induction on $k \in \mathbb{Z}$ with $1 \leq k \leq |C| - 1$ that there exists an edge $e \in C \setminus C_k$ that is contained in an odd hole of $G_k := G_0 - C_k$.

In case $k = 1$, the edge e_0 is critical in $G_0 - (C \setminus \{e\})$ and therefore contained in an odd hole of $G_0 - (C \setminus \{e_0\}) = G_0 - C_1 = G_1$.

Suppose the claim holds for some fixed $k \in \mathbb{Z}$ with $1 \leq k \leq |C|$ and let $e \in C \setminus C_k$ be an edge that is contained in an hole $H = (V_H, E_H)$ in G_k . We will proof that the claim also holds for $k + 1$. When adding edge e_k to the graph $G_k = G_0 - C_k$, the following cases can occur:

1. $|e \cap V_H| \leq 1$: In this case H is also an odd hole in $G_k + \{e\} = G_0 - (C_k \setminus \{e\}) = G_{k+1}$.
2. $|e \cap V_H| = 2$: In this case e is a chord of the odd cycle H in $G_k + \{e\} = G_0 - (C_k \setminus \{e\}) = G_{k+1}$ and there exists an odd hole H^+ in G_{k+1} such that $e \in E(H^+)$ and $E(H^+) \subsetneq E_H \cup \{e\}$.

By induction this proves the claim for $k \in \mathbb{Z}$ with $1 \leq k \leq |C|$. For $k = |C|$ we obtain the statement of the theorem. \square

We have seen that odd holes and critical (sets of) edges in facet-graphs are strongly linked. The following lemma extends this link even more.

Lemma 4. *Let $G_0 = (V_0, E_0)$ with weights $\pi_v > 0$ for all $v \in V_0$ be a connected facet-graph with $|V_0| \geq 3$. Then there exists a spanning α_π -critical subgraph T with $\alpha_\pi(G_0) = \alpha_\pi(T)$ such that every edge $e \in E(T)$ is contained in some odd hole H_e in G_0 , i.e., $e \in E(H_e)$.*

Proof. Suppose we create a spanning α_π -critical subgraph T of G_0 as follows. First, non-critical edges that are not contained in an odd hole in G_0 are iteratively deleted. Afterwards, all other non-critical edges are iteratively deleted.

Assume there exists an edge $e \in E(T)$ that is not part of an odd hole in G_0 . Let $\bar{E} \subseteq E(G_0) \setminus E(T)$ be minimal such that e is critical in $G_0 - \bar{E}$. Then $\bar{E} \cup \{e\}$ is a minimal critical set of edges in G_0 and by Lem. 3 there exists an edge $e_h \in \bar{E} \cup \{e\}$ that is part of an odd hole in G_0 . Before e_h was removed from G_0 both edges e and e_h were not critical and hence, we would have deleted the edge e before edge e_h . This is a contradiction to the fact that e_h was removed before e . \square

With Lem. 4 we are finally ready to prove one of our main results.

Theorem 3. *Let $G = (V, E)$ be a graph with $n := |V|$, let $E' \subseteq E$ be a set of the edges, and define $G' := (V, E')$. If G' contains all odd holes of G , then $DW(G, G') = \text{STAB}(G)$ holds.*

Proof. Let $\sum_{v \in V} \pi_v x_v \leq \pi_0$ be some non-trivial, non-edge facet of $\text{STAB}(G)$. Let $V_0 := \{v \in V : \pi_v > 0\}$ and let $G_0 := G[V_0] = (V_0, E_0)$ be the induced subgraph on the nodes with positive weights. Then G_0 with weights $\pi_v > 0$ for all $v \in V_0$ is a facet graph and $|V_0| \geq 3$ holds.

Since every critical edge $e \in E_0$ is contained in some odd hole $H = (V_H, E_H)$ in G_0 and G_0 is an induced subgraph of G , every odd hole H is also an odd hole in G . Let T_0 be a spanning α_π -critical subgraph of G_0 such that every edge $e \in E(T_0)$ is contained in some odd hole H_e in G_0 . Such a graph exists due to Lem. 4. Hence, all edges of T_0 are contained in E' and T_0 is a subgraph of G' .

The inequality $\sum_{v \in V_0} \pi_v x_v \leq \pi_0$ is valid for $\text{STAB}(T_0)$ and since T_0 is a subgraph of G' , the inequality is also valid for $\text{STAB}(G')$. Hence, the inequality $\sum_{v \in V_0} \pi_v x_v \leq \pi_0$ is also valid for $\text{DW}(G, G') \subseteq \text{STAB}(G')$

The facet $\sum_{v \in V} \pi_v x_v \leq \pi_0$ was chosen arbitrarily, which implies that $\text{DW}(G, G') \subseteq \text{STAB}(G)$ and therefore $\text{DW}(G, G') = \text{STAB}(G)$ holds. \square

Together with Thm. 2 this gives us a characterization of the strongest possible Dantzig-Wolfe reformulations.

Theorem 4. *Let $G = (V, E)$ be a graph with $n := |V|$, let $E' \subseteq E$ be a subset of the edges, and define $G' := (V, E')$. Then $\text{DW}(G, G') = \text{STAB}(G)$ if and only if G' contains all odd holes of G .*

Proof. Follows directly from Thm. 2 and 3. \square

5 Discussion

Dantzig-Wolfe reformulation of an integer program naturally lends itself to a multitude of relaxations since we are free to choose the decomposition to apply to a model. In particular, we always have the (uninteresting) options to reformulate *all* constraints, thereby obtaining the integer hull, or to reformulate *no* constraint, leading to the LP relaxation. Consequently, asking for “the” strength of Dantzig-Wolfe reformulation is non-trivial in general. As a seminal step towards an answer we focus on the textbook model of the stable set problem. It has been the object of study in previous investigations of the strength of several other relaxations.

In this paper we characterized the two extreme cases. We have seen that we obtain a weakest possible Dantzig-Wolfe reformulation if and only if the convexified subgraph is bipartite. A Dantzig-Wolfe reformulation that is strongest possible is obtained if and only if the convexified subgraph contains all odd holes of the original graph.

In order to derive the characterization of strongest possible Dantzig-Wolfe reformulations, we investigated the structure of facet-graphs and proved some new results. Sewell asked in his Phd thesis [21] if every pair of critical edges in a facet-graph is contained in an odd hole. We reconsidered this question and proved that every critical edge in a facet-graph is contained in an odd hole. In addition, we generalized this result to critical sets of edges. In the end, we proved that every facet-graph contains a spanning α_π -critical subgraph such that every

edge of the spanning critical subgraph is contained in an odd hole of the initial facet-graph. This result has an impact on the separation of cutting planes. When searching for facet-inducing cutting planes, we only have to consider the subgraph spanned by all edges contained in some odd hole of the graph.

Dealing with computational implications of our work is out of the scope of this paper. If we wanted to apply the strongest possible Dantzig-Wolfe reformulation in practice, we had to decide which edges are contained in an odd hole. Unfortunately, deciding this for a given edge is an NP-complete problem [5] already. Moreover, the parameterized problem of deciding whether a given edge is contained in an odd hole of size at most k for some given parameter $k \in \mathbb{Z}_{\geq 3}$ is W[1]-complete [13]. Even if we managed to practically produce the reformulation, we expect the LP relaxation of the Dantzig-Wolfe reformulation to be too hard to solve effectively in practice. We will report on experiments in future work.

From a theoretical point of view, a logical next step is to investigate Dantzig-Wolfe reformulations between the two extremes. An interesting family of reformulations is obtained by convexifying all holes of size at most $k \in \mathbb{Z}_{\geq 0}$, which may be fixed or dependant on the size of the graph. This may yield a provably strong formulation, while solving the LP relaxation of the Dantzig-Wolfe reformulation stays computationally tractable.

The most interesting question is how our work can be generalized. We believe that the characterizations of weakest and strongest possible Dantzig-Wolfe reformulations can be directly extended to intimately related problems like the node covering problem or the linear ordering problem. It is our long-term vision to concentrate on more general integer programs. We hope that our work contributes to an understanding of “the” strength of Dantzig-Wolfe reformulations in general.

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