Solving Bulk-Robust Assignment Problems to Optimality

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Joint work with

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Assignment Problem:

- Input: Bipartite graph $G = (V, E)$ with $V = A \mathbin{\dot\cup} B$, edge costs $c \in \mathbb{R}^E$
- Feasible sets: Perfect matchings $M \subseteq E$ (assuming $|A| = |B|$)
- Goal: Minimize cost $c(M) := \sum_{e \in M} c_e$

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Bulk-Robustness:

- Possible (or likely) failure scenarios are given (explicitly or implicitly).
- Goal: Buy edges such that for every scenario, there still exists a perfect matching using the (bought) edges that survived.

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Literature:

- Concept formally introduced by Adjiashvili, Stiller & Zenklusen (MPA 2015)
- Classical related problems: k-edge connected spanning subgraph problem robustifies spanning-tree problem against failure of any $(k-1)$-edge set.

Bulk-Robust Assignments with Edge Failures

Input:

- Bipartite graph $G = (V, E)$ with $V = A \cupdot B$
- Failure scenarios $\mathcal{F} = \{\{f_1\}, \dots, \{f_\ell\}\}$ with $f_i \in E$.
- Edge costs $c \in \mathbb{R}^E$

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Goal:

- Find $X \subseteq E$ with minimum $c(X)$ such that
- for all $F \in \mathcal{F}$, the subgraph $(V, X \setminus F)$ contains a perfect matching.

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Algorithm by Adjiashvili, Bindewald & Michaels (ICALP 2016):

- Approximation factor: $\mathcal{O}(\log |V|)$
- Outline:
  1. Solve LP relaxation of IP formulation.
  2. Decompose LP optimum of some failure-specific part into convex combination of perfect matchings.
  3. Randomly select one such matching $\bar{M}$ according to decomposition distribution.
  4. Augment current solution by edges that improve connectivity.

Bulk-Robust Assignments with Node Failures

Input:

- Bipartite graph $G = (V, E)$ with $V = A \dot\cup B$
- Failure scenarios $\mathcal{F} = \{\{b_1\}, \ldots, \{b_\ell\}\}$ with $b_i \in B$.
- Node costs $c \in \mathbb{R}^B$

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- Failure scenarios $\mathcal{F} = \{\{b_1\}, \ldots, \{b_\ell\}\}$ with $b_i \in B$.
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Goal:

- Find $X \subseteq B$ with minimum $c(X)$ such that
- for all $F \in \mathcal{F}$, the subgraph $G[A \cup X \setminus F]$ contains an A-perfect matching (a matching that covers A).

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Algorithm by Adjiashvili, Bindewald & Michaels (2017):

- Approximation factor: $\log |A| + 2$

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Algorithm by Adjiashvili, Bindewald & Michaels (2017):

- Approximation factor: $\log|A| + 2$
- Outline:
  1. Compute an A-perfect matching w.r.t. certain costs.
  2. Reduce the remaining problem to a set-cover instance.
  3. Solve the latter by the greedy algorithm.

Generalization

Input:

- Bipartite graph $G = (V, E)$ with $V = A \dot\cup B$
- Failure scenarios $\mathcal{F} = \{F_1, \dots, F_\ell\}$ with $F_i \subseteq E$ with cardinalities $k(F)$ for all $F \in \mathcal{F}$
- Edge costs $c \in \mathbb{R}^E$

Goal:

- Find $X \subseteq E$ with minimum $c(X)$ such that
- for all $F \in \mathcal{F}$, the subgraph $(V, X \setminus F)$ contains a matching of size $k(F)$.

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Special cases:

- Edge failures: Set $k(F_i) := |A| = |B|$ and $F_i := \{f_i\}$ for all $i \in [\ell]$.
- Node failures: Set $k(F_i) := |A|$ and $F_i := \delta(b_i)$ for all $i \in [\ell]$.

Straight-forward model (see Adjiashvili et al., ICALP 2016):

$$
\begin{aligned}
\min \quad & c^{\top} x & & \\
\text{s.t.} \quad & x \geq y^{(F)} & \text{for all } F \in \mathcal{F} & \quad (1)\\
& y^{(F)} \in P_{k(F)\text{-match}}(G - F) & \text{for all } F \in \mathcal{F} & \quad (2)\\
& x_e \in \mathbb{Z}_+ & \text{for all } e \in E & \quad (3)
\end{aligned}
$$

- Has $\mathcal{O}(|\mathcal{F}| \cdot |E|)$ variables and constraints.

Integer Programming Models

Straight-forward model (see Adjiashvili et al., ICALP 2016):

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Polyhedral combinatorics helps:

- What does this mean for x?

$$\exists y : x \geq y,\ y \in P_{k(F)\text{-match}}(G')$$

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$$\exists y : x \geq y, \; y \in P_{k(F)\text{-match}}(G')$$

- Projection onto x is the dominant of the $k(F)$-matching polytope.

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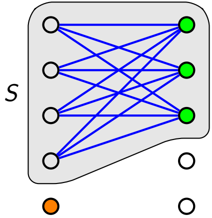
Polyhedral combinatorics helps:

- What does this mean for x?

$$\exists y : x \geq y, \; y \in P_{k(F)\text{-match}}(G')$$

- Projection onto x is the dominant of the $k(F)$-matching polytope.
- Inequalities known (Fulkerson '70):

$$\sum_{e \in E[S]} x_e \geq |S| - |V| + k(F) \quad \text{for all } S \subseteq V$$



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$$

- Has $\mathcal{O}(|\mathcal{F}| \cdot |E|)$ variables and constraints.

Equivalent (derived from dominant):

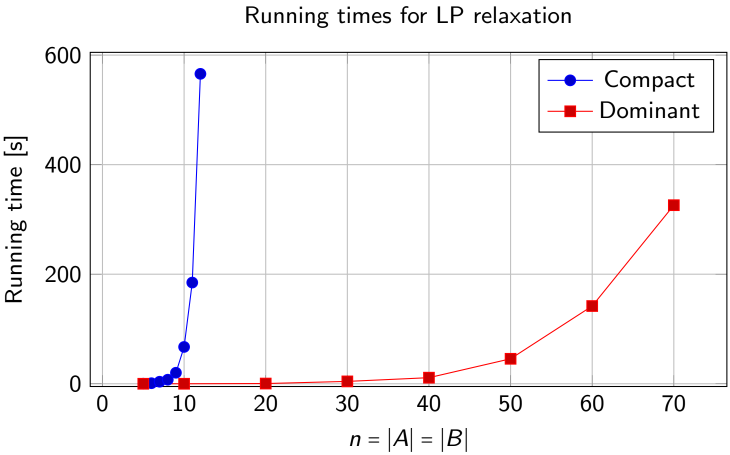
$$
\begin{aligned}
\min\ & c^\top x & & \\
\text{s.t.}\ & \sum_{e \in E[S] \setminus F} x_e \geq |S| - |V| + k(F) & \text{for all } S \subseteq V \text{ for all } F \in \mathcal{F} & \quad (4)\\
& x_e \in \mathbb{Z}_+ & \text{for all } e \in E & \quad (5)
\end{aligned}
$$

- Has $\mathcal{O}(|E|)$ variables and $\mathcal{O}\left(|\mathcal{F}| \cdot 2^{|V|}\right)$ constraints.
- For every $F \in \mathcal{F}$, separation problem reduces to a minimum-cut problem.

Models in Practice: LP Relaxation

Setup:

- Complete bipartite graphs with $|A| = |B| = n$
- Uniform failures $\mathcal{F} = \{\{e\} \mid e \in E\}$
- Unit costs $c = \mathbb{1}$
- Time limit 600 s



Models in Practice: LP Relaxation

Setup:

- Erdős-Rényi graphs with $|A| = |B| = n$, $p = 0.4$
- Uniform failures $\mathcal{F} = \{\{e\} \mid e \in E\}$
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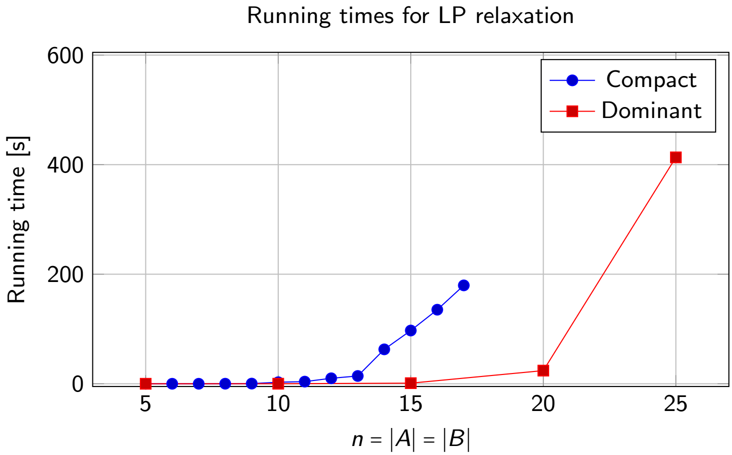

Models in Practice: IP Bounds

Setup:

- Complete bipartite graphs with $|A| = |B| = n$
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- Time limit 600 s

Results for compact model

n	Root bound	Root time [s]	Final bound	Time [s]	Optimum
5	6.25	0.1	10.00	5.8	10
6	7.20	1.0	9.73	600	12
7	8.17	3.7	9.20	600	14
8	9.29	7.4	9.86	600	16
9	10.13	20.1	10.29	600	18
10	11.11	67.2	11.25	600	20
11	12.10	184.74	12.10	600	22

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Results for dominant model

n	Root bound	Root time [s]	Final bound	Time [s]	Optimum
5	6.25	0.0	10.00	0.2	10
6	7.20	0.0	12.00	4.6	12
7	8.17	0.0	14.00	179.5	14
8	9.29	0.0	13.00	600	16
9	10.13	0.0	13.45	600	18
10	11.11	0.0	13.92	600	20
11	12.10	0.0	14.40	600	22

Solving with Gurobi

Setup:

- Combination of complete graph, singleton failures and unit costs has lots of symmetry.
- Gurobi detects this and can prove lower bound earlier.
- For $n = 9$, we observe this:

Results for compact model with Gurobi

```
    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0    9.14286    0 3648   64.00000    9.14286  85.7%     -    4s
     0     0    9.14286    0 3648   64.00000    9.14286  85.7%     -    6s
     0     2    9.14286    0 3648   64.00000    9.14286  85.7%     -   13s
     2     2   10.00000    1 3521   64.00000   10.00000  84.4%  5738   15s
...
   525   355   13.00000   16 1267   16.00000   11.73810  26.6%  1526  205s
   542   349 infeasible   16        16.00000   13.42857  16.1%  1509  210s
   548   352   13.80000   19 2011   16.00000   13.80000  13.7%  1519  215s
   559   345   15.00000   19 1923   16.00000   14.00000  12.5%  1525  220s

Explored 568 nodes (899200 simplex iterations) in 223.95 seconds
Thread count was 1 (of 4 available processors)

Optimal solution found (tolerance 1.00e-04)
Best objective 1.600000000000e+01, best bound 1.600000000000e+01, gap 0.0%
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Note: This effect vanishes as soon as the graph is not symmetric anymore.

Strengthening the Model

Chvátal-Gomory cuts:

- Consider $F_1, \dots, F_\ell$ with constant $k(F_i) = k$ for all $i \in [\ell]$ $(\ell \geq 2)$.
- Sum up all inequalities for fixed S with $|S| - |V| + k \geq 1$.

Strengthening the Model

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$$\sum_{e \in E[S]} |\{i \in [\ell] \mid e \notin F_i\}| x_e \geq \ell(|S| - |V| + k)$$

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- Scale it by $1/(\ell - 1)$.

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$$\sum_{e \in E[S]} \frac{|\{i \in [\ell] \mid e \notin F_i\}|}{\ell - 1} x_e \geq \frac{\ell}{\ell - 1}(|S| - |V| + k)$$

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- x is integer and nonnegative, so round up coefficients and right-hand side.

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- x is integer and nonnegative, so round up coefficients and right-hand side.

$$\sum_{e \in E[S]} \left\{ \begin{array}{ll} 2 & \text{if } e \text{ in no } F_i \\ 0 & \text{if } e \text{ in all } F_i \\ 1 & \text{otherwise} \end{array} \right\} x_e \geq |S| - |V| + k + 1$$

Strengthening the Model

Chvátal-Gomory cuts:

- Consider $F_1, \dots, F_\ell$ with constant $k(F_i) = k$ for all $i \in [\ell]$ ($\ell \geq 2$).
- Sum up all inequalities for fixed S with $|S| - |V| + k \geq 1$.

$$\sum_{e \in E[S]} |\{i \in [\ell] \mid e \notin F_i\}| x_e \geq \ell(|S| - |V| + k)$$

- Scale it by $1/(\ell - 1)$.

$$\sum_{e \in E[S]} \frac{|\{i \in [\ell] \mid e \notin F_i\}|}{\ell - 1} x_e \geq \frac{\ell}{\ell - 1}(|S| - |V| + k)$$

- x is integer and nonnegative, so round up coefficients and right-hand side.

$$\sum_{e \in E[S]} \begin{Bmatrix} 2 & \text{if } e \text{ in no } F_i \\ 0 & \text{if } e \text{ in all } F_i \\ 1 & \text{otherwise} \end{Bmatrix} x_e \geq |S| - |V| + k + 1$$

- Weakened for coefficients with e in no F_i.
- Strengthened for coefficients with e in all F_i.
- Stronger right-hand side.

Separation Problem

Input:

- Bipartite graph $G = (V, E)$ with bipartition $V = A \mathbin{\dot\cup} B$.
- Edge weights $w \in \mathbb{R}_+^E$
- Parameter k.

Goal:

- Find $S \subseteq V$ with $|S| \geq |V| - k + 1$ minimizing $w(E[S]) - |S| + |V| - k$

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IP Model:

- Variables y and z with
- $y_v = 1 \iff v \in S$
- $z_e = 1 \iff e \in E[S]$

$$
\begin{aligned}
\min \quad & -\sum_{v \in V} y_v + \sum_{e \in E} w_e z_e \\
\text{s.t.} \quad & -y_a - y_b + z_{a,b} \geq -1 \quad \text{for all } \{a, b\} \in E \\
& y(A) + y(B) \geq |V| - k + 1 \\
& y, z \text{ binary}
\end{aligned}
$$

Observe: TU system plus a single inequality.

Separation Problem

Relation to the dominant separation problem:

- Find $S \subseteq V$ with $|S| \geq |V| - k$ minimizing $w(E[S]) - |S| + |V| - k$
- For $k = \frac{1}{2}|V|$ (perfect matchings), $|S| \geq \frac{1}{2}|V|$ can be ignored via

$$S := A \qquad \text{and} \qquad E[S] = \emptyset$$

Models in Practice: IP Bounds with CG Cuts

Setup:

- Complete bipartite graphs with $|A| = |B| = n$
- Uniform failures $\mathcal{F} = \{\{e\} \mid e \in E\}$
- Unit costs $c = \mathbb{1}$
- Time limit 600 s

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Results for dominant / CG model

n	Dominant		+CG			Optimum
5	10.00	0.2 s	10.00	0.4 s		10
6	12.00	4.6 s	12.00	0.9 s		12
7	14.00	179.5 s	14.00	2.7 s		14
8	13.00	600.0 s	16.00	11.3 s		16
9	13.45	600.0 s	18.00	26.5 s		18
10	13.92	600.0 s	20.00	92.8 s		20
11	14.40	600.0 s	22.00	332.1 s		22

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- Uniform failures $\mathcal{F} = \{\{e\} \mid e \in E\}$
- Unit costs $c = \mathbb{1}$
- Time limit 600 s
- Special case of CG cuts are strengthened degree inequalities $x(\delta(v)) \geq 2$.
- These already prove a dual bound of $2n$. Let's add those in the beginning!

Results for dominant / CG model

n	Dominant		+CG		+Initial Degree		Optimum
5	10.00	0.2 s	10.00	0.4 s	10.00	0.0 s	10
6	12.00	4.6 s	12.00	0.9 s	12.00	0.0 s	12
7	14.00	179.5 s	14.00	2.7 s	14.00	0.0 s	14
8	13.00	600.0 s	16.00	11.3 s	16.00	0.0 s	16
9	13.45	600.0 s	18.00	26.5 s	18.00	0.0 s	18
10	13.92	600.0 s	20.00	92.8 s	20.00	0.0 s	20
11	14.40	600.0 s	22.00	332.1 s	22.00	0.1 s	22

Models in Practice: IP Bounds with CG Cuts

Setup:

- Erdős-Rényi graphs with $|A| = |B| = n$, $p = 0.4$
- Uniform failures $\mathcal{F} = \{\{e\} \mid e \in E\}$
- Unit costs $c = \mathbb{1}$
- Time limit 600 s

Results for dominant / CG model

n	+CG		+Initial Degree		Optimum
5	10.00	0.4 s	10.00	0.0 s	10
6	12.00	1.0 s	12.00	0.0 s	12
7	14.00	2.9 s	14.00	0.0 s	14
8	16.00	11.8 s	16.00	0.0 s	16
9	18.00	27.2 s	18.00	0.0 s	18
10	20.00	3.3 s	20.00	0.0 s	20
11	22.00	8.9 s	22.00	0.0 s	22
12	22.00	22.6 s	22.00	23.6 s	22

Models in Practice: IP Bounds with CG Cuts

Setup:

- Erdős-Rényi graphs with $|A| = |B| = n$, $p = 0.4$
- Uniform failures $\mathcal{F} = \{\{e\} \mid e \in E\}$
- Unit costs $c = \mathbb{1}$
- Time limit 600 s

Results for dominant / CG model

n	+CG		+Initial Degree		Optimum
5	10.00	0.4 s	10.00	0.0 s	10
6	12.00	1.0 s	12.00	0.0 s	12
7	14.00	2.9 s	14.00	0.0 s	14
8	16.00	11.8 s	16.00	0.0 s	16
9	18.00	27.2 s	18.00	0.0 s	18
10	20.00	3.3 s	20.00	0.0 s	20
11	22.00	8.9 s	22.00	0.0 s	22
12	22.00	22.6 s	22.00	23.6 s	22

Observations:

- Have to average over many instances to get authoritative statistics.
- No strengthened degree constraints if less than two scenarios F with $k(F) = \frac{1}{2}|V|$.

Models in Practice: IP Bounds with CG Cuts

Setup:

- Complete bipartite graphs with $|A| = |B| = n$
- Uniform failures $\mathcal{F} = \{\{e\} \mid e \in E\}$
- Unit costs, random $c_e \in \{1, 2, 3\}$ for all $e \in E$ independently.
- Time limit 600 s

Results for dominant / CG model

n	Dominant+CG		+Initial Degree		Optimum
5	12.00	0.2 s	12.00	0.0 s	12
6	15.00	0.7 s	15.00	0.0 s	15
7	17.00	1.4 s	17.00	0.0 s	17
8	20.00	3.9 s	20.00	0.0 s	20
9	20.00	8.9 s	20.00	0.0 s	20
10	21.00	15.0 s	21.00	0.0 s	21
11	22.00	44.6 s	22.00	0.0 s	22
12	29.00	128.1 s	29.00	0.0 s	29
13	27.00	303.3 s	27.00	0.0 s	27

Observations:

- Random costs make reduce solution times.

Models in Practice: IP Bounds with CG Cuts

Setup:

- Complete bipartite graphs with $|A| = n$ and $|B| = \lfloor 1.5n \rfloor$
- Node failures $\mathcal{F} = \{\delta(b) \mid b \in B\}$
- Unit costs $c = \mathbb{1}$
- Time limit 600 s

Results for dominant / CG model

n	Dominant+CG		+Initial Degree		Optimum
5	10.00	1.0 s	10.00	0.1 s	10
6	12.00	2.6 s	12.00	0.1 s	12
7	14.00	6.8 s	14.00	0.4 s	14
8	16.00	60.5 s	16.00	7.0 s	16
9	18.00	297.3 s	18.00	5.1 s	18
10	$\geq$ 11.00	600.0 s	20.00	0.5 s	20
11	$\geq$ 12.00	600.0 s	22.00	21.0 s	22
13	$\geq$ 13.00	600.0 s	24.00	34.0 s	24
14	$\geq$ 14.00	600.0 s	26.00	41.4 s	26
15	$\geq$ 15.00	600.0 s	28.00	25.7 s	28
15	$\geq$ 16.00	600.0 s	30.00	46.4 s	30

Observations:

- Initial degree constraints are again very strong.
- Time mostly used for primal bound!

Thanks!

Things you've seen:

- Speed-up of dominant formulation vs. compact one.
- Derivation of Chvátal-Gomory (CG) cuts.
- Strength of CG cuts, in particular strengthened degree.

Things you might see in the future:

- Even faster code to exploit different strengths of CG cuts.
- More (structured) instances
- More experiments (averaging over random instance)