Compact quadratizations for pseudo-Boolean functions

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joint work with
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Pseudo-Boolean optimization

General problem: pseudo-Boolean optimization

Given a pseudo-Boolean function $f : \{0,1\}^n \to \mathbb{R}$

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Every pseudo-Boolean function $f : \{0,1\}^n \to \mathbb{R}$ admits a unique multilinear expression.

- Given f, finding its unique multilinear representation can be costly! (Size of the input: $O(2^n)$)

Multilinear 0–1 optimization

Assumption: f given as a multilinear polynomial

Set of monomials $\mathcal{S} \subseteq 2^{[n]}$, $a_S \neq 0$ for $S \in \mathcal{S}$.

$$\begin{aligned} \min & \sum_{S \in \mathcal{S}} a_S \prod_{i \in S} x_i \\ \text{s. t. } & x_i \in \{0, 1\}, \text{ for } i = 1, \dots, n \end{aligned}$$

Example:

$$f(x_1, x_2, x_3) = 9x_1x_2x_3 + 8x_1x_2 - 6x_2x_3 + x_1 - 2x_2 + x_3$$

Quadratization: definition and desirable properties

Definition (Anthony, Boros, Crama, & Gruber, 2017)

Given a pseudo-Boolean function $f(x)$ where $x \in \{0,1\}^n$, a *quadratization* $g(x, y)$ is a function satisfying

- g is quadratic
- $g(x, y)$ depends on the original variables x and on m auxiliary variables y
- satisfies

$$f(x) = \min\{g(x, y) : y \in \{0,1\}^m\} \quad \forall x \in \{0,1\}^n.$$

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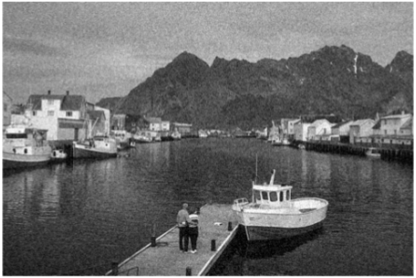
Which quadratizations are "good"?

- Small number of auxiliary variables (*compact*).
- Small number of positive quadratic terms ($x_i x_j$, $x_i y_j \dots$) (empirical measure of submodularity).
- ...

Application in computer vision: image restoration

Input: blurred image



Output: restored image

Image from the Corel database.

Persistencies

Weak Persistency Theorem (Hammer, Hansen, & Simeone, 1984)

Let (QP) be a quadratic optimization problem on $x \in \{0,1\}^n$, and let $(\tilde{x}, \tilde{y})$ be an optimal solution of the *continuous standard linearization of* (QP)

$$
\begin{aligned}
\min \quad & c_0 + \sum_{j=1}^{n} c_j x_j + \sum_{1 \le i < j \le n} c_{ij} y_{ij} & \\
\text{s. t.} \quad & y_{ij} \ge x_i + x_j - 1 & i, j = 1, \dots, n, i < j \\
& y_{ij} \le x_i & i, j = 1, \dots, n, i < j \\
& y_{ij} \le x_j & i, j = 1, \dots, n, i < j \\
& 0 \le y_{ij} \le 1 & i, j = 1, \dots, n, i < j \\
& 0 \le x_i \le 1 & i = 1, \dots, n
\end{aligned}
$$

such that $\tilde{x}_j = 1$ for $j \in O$ and $\tilde{x}_j = 0$ for $j \in Z$. Then, for any minimizing vector x^* of (QP) switching $x_j^* = 1$ for $j \in O$ and $x_j^* = 0$ for $j \in Z$ will also yield a minimum of f.

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- There are ways to compute, in polynomial time, a *maximal* set of variables to fix, based on a network flow algorithm (Boros et al., 2008).
- In computer vision, image restoration and related problems of up to *millions* of variables are efficiently solved, thanks to the use of persistencies.

Termwise quadratizations

Main idea

Quadratize monomial by monomial using disjoint sets of auxiliary variables.

$$f(x) = -35x_1x_2x_3x_4x_5 + 50x_1x_2x_3x_4 - 10x_1x_2x_4x_5 + 5x_2x_3x_4 + 5x_4x_5 - 20x_1$$

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Negative monomial

(Kolmogorov & Zabih, 2004; Freedman & Drineas, 2005)

$$-\prod_{i=1}^{n} x_i = \min_{y \in \{0,1\}} -y\left(\sum_{i=1}^{n} x_i - (n-1)\right)$$

- One variable is sufficient.
- No positive quadratic terms.

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1. If $x_i = 1\ \forall i$, then $min_y - y$, minimum value of -1 reached for $y = 1$.
2. If $\exists i$ such that $x_i = 0$, then $min_y - Cy$, where $C \leq 0$, minimum value of 0 reached for $y = 0$.

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Positive monomial

(Ishikawa, 2011)

$$\prod_{i=1}^{n} x_i = \min_{y\in\{0,1\}^k} \sum_{i=1}^{k} y_i(c_{i,n}(-|x| + 2i) - 1) + \frac{|x|(|x|-1)}{2},$$

- Number of variables: $k = \lfloor \frac{n-1}{2} \rfloor$.
- $\binom{n}{2}$ positive quadratic terms.

Upper bound for the positive monomial: $\lceil \log(n) \rceil - 1$

Theorem 3 (simplified version)

Assume that $n = 2^\ell$ and let $|x| = \sum_{i=1}^n x_i$ be the Hamming weight of $x \in \{0,1\}^n$. Then,

$$g(x,y) = \frac{1}{2}(|x| - \sum_{i=1}^{\ell-1} 2^i y_i)(|x| - \sum_{i=1}^{\ell-1} 2^i y_i - 1)$$

is a quadratization of the positive monomial $P_n(x) = \prod_{i=1}^n x_i$ using $\lceil \log(n) \rceil - 1$ auxiliary variables.

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- For $|x| \leq n-1$, one factor to reach the minimum value of zero for *odd* $|x|$ and the other factor for *even* $|x|$.

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- For $|x| \leq n-1$, one factor to reach the minimum value of zero for *odd* $|x|$ and the other factor for *even* $|x|$.
- Case 2 ($|x| = n$): Similarly, we can show $min_y g(x,y) = 1$.

Lower bound for the positive monomial

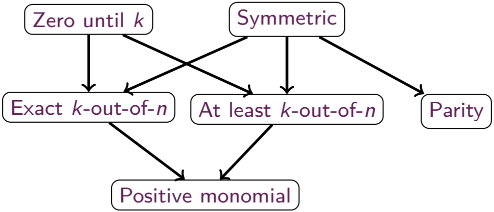
Theorem 3

If $g(x, y)$ is a quadratization of the positive monomial $P_n(x) = \prod_{i=1}^{n} x_i$ using m variables, then

$$m \geq \lceil \log(n) \rceil - 1$$

Results for more general functions

Function	Lower Bound	Upper Bound
Zero until k	$\Omega(2^{\frac{n}{2}})$ for some function[1] $\lceil \log(k) \rceil - 1$ for all functions	$O(2^{\frac{n}{2}})$ [1]
Symmetric	$\Omega(\sqrt{n})$ for some function[2]	$O(\sqrt{n}) = 2\lceil \sqrt{n+1} \rceil$
Exact k-out-of-n	$\max(\lceil \log(k) \rceil, \lceil \log(n-k) \rceil) - 1$	$\max(\lceil \log(k) \rceil, \lceil \log(n-k) \rceil)$
At least k-out-of-n	$\lceil \log(k) \rceil - 1$	$\max(\lceil \log(k) \rceil, \lceil \log(n-k) \rceil)$
Positive monomial	$\lceil \log(n) \rceil - 1$	$\lceil \log(n) \rceil - 1$
Parity	$\lceil \log(n) \rceil - 1$	$\lceil \log(n) \rceil - 1$



[1] see (Anthony et al., 2017)

[2] see (Anthony, Boros, Crama, & Gruber, 2016)

Ongoing computational work

Which quadratizations work better in practice?

Pairwise covers

Anthony, Boros, Crama and Gruber (2017)

Substituting common sets of variables

$$f(x) = -35x_1x_2x_3x_4x_5 + 50x_1x_2x_3x_4 - 10x_1x_2x_4x_5 + 5x_2x_3x_4 + 5x_4x_5 - 20x_1$$

could be replaced by

$$f(x) = -35y_{12}y_{345} + 50y_{12}y_{34} - 10y_{12}y_{45} + 5x_2y_{34} + 5x_4x_5 - 20x_1 + P(x,y)$$

where $P(x,y)$ imposes $y_{12} = x_1x_2$, $y_{345} = y_{34}x_5$...

Heuristics for small Pairwise Covers

Three heuristics:

- PC1: Separate first two variables from the rest.
- PC2: Most “popular” intersections first.
- PC3: Most “popular” pairs of variables first.

Main idea: identifying *subterms* that appear as subsets of one or more monomials in the input monomial set $\mathcal{S}$.

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- ... but we already obtain some interesting observations.
- We compare the results with the resolution of linearized instances (SL) using CPLEX 12.7.

Instances: Vision

Image restoration

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \longrightarrow \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Base images:

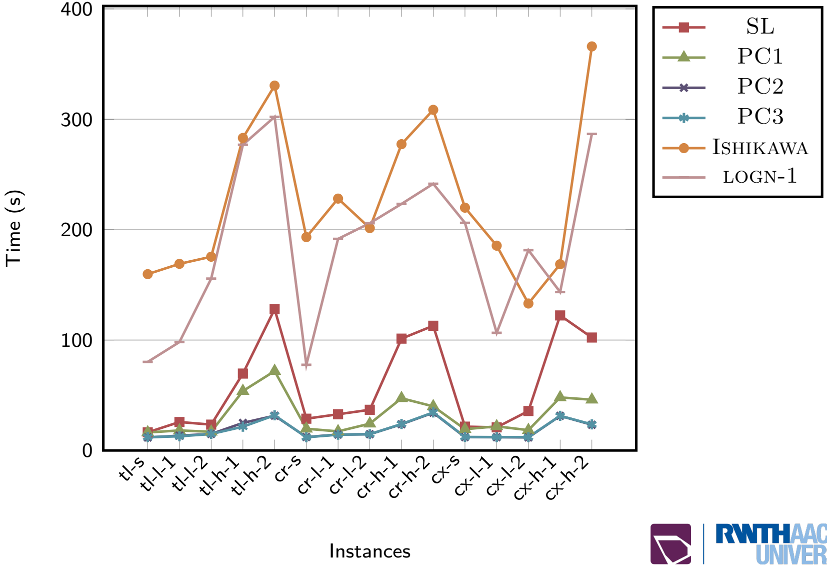
- top left rect. (tl)
- centre rect. (cr)
- cross (cx)

Perturbations:

- none (n)
- low (l)
- high (h)

Up to $n = 900$ variables and $m = 6788$ terms

Vision: all methods 15 × 15 ($n = 225$, $m = 1598$)



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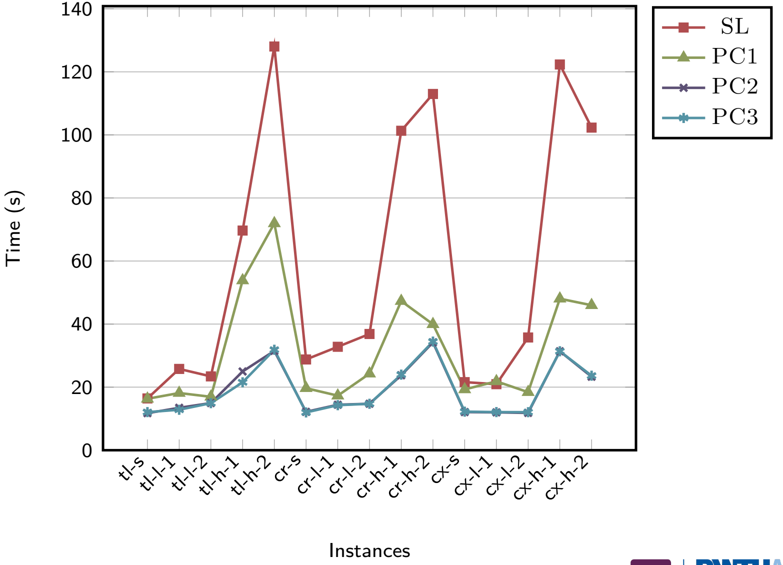
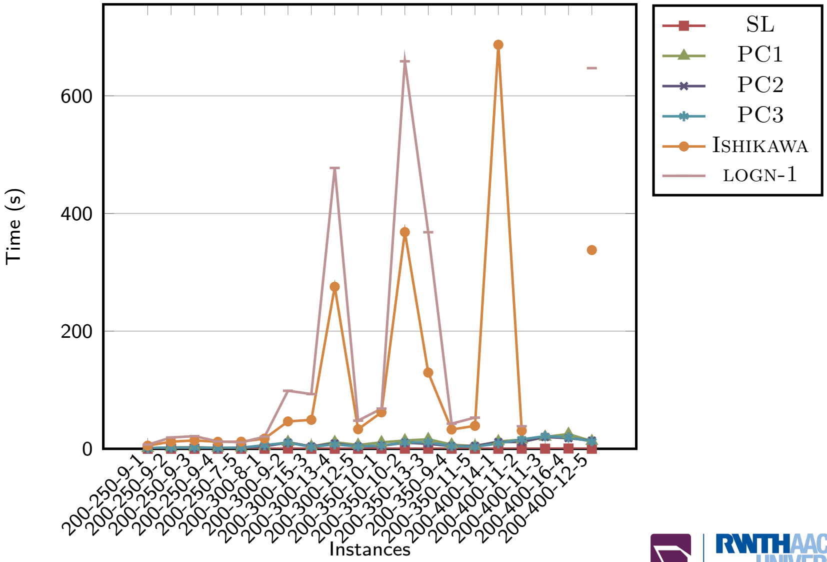


Random polynomials: all methods



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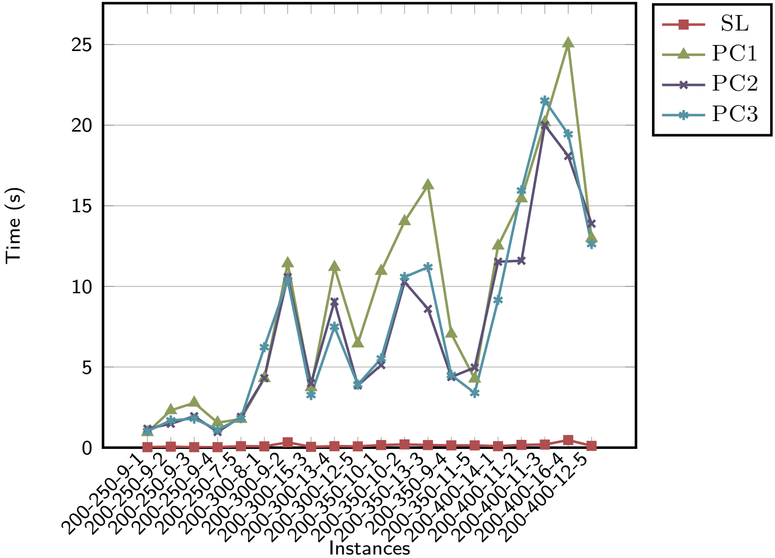

Quadratization properties

Vision	Pairwise covers	Termwise
Number of y variables	less	more
Number of positive quadratic terms	less	more

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Number of y variables	**more**	**less**
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Summary

- New compact quadratizations for the positive monomial.
- Proof of the lower bound on the number of auxiliary variables.
- First experiments: small number of auxiliary variables might not be the best criterion to define good quadratizations.

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Perspectives

- Experiments will be re-tested using persistencies and other solvers.
- Other properties:
  - Small number of positive quadratic terms.
  - Graph underlying quadratic terms with special structure (e. g. sparse...).
  - ...

Bibliography I

Anthony, M., Boros, E., Crama, Y., & Gruber, A. (2016). Quadratization of symmetric pseudo-Boolean functions. *Discrete Applied Mathematics*, *203*, 1 – 12.

Anthony, M., Boros, E., Crama, Y., & Gruber, A. (2017). Quadratic reformulations of nonlinear binary optimization problems. *Mathematical Programming*, *162*(1-2), 115–144.

Boros, E., Hammer, P. L., Sun, X., & Tavares, G. (2008). A max-flow approach to improved lower bounds for quadratic unconstrained binary optimization (QUBO). *Discrete Optimization*, *5*(2), 501–529. (In Memory of George B. Dantzig)

Freedman, D., & Drineas, P. (2005, June). Energy minimization via graph cuts: settling what is possible. In *Ieee conference on computer vision and pattern recognition* (Vol. 2, pp. 939–946).

Bibliography II

Hammer, P. L., Hansen, P., & Simeone, B. (1984). Roof duality, complementation and persistency in quadratic 0—1 optimization. *Mathematical Programming*, *28*(2), 121–155.

Hammer, P. L., Rosenberg, I., & Rudeanu, S. (1963). On the determination of the minima of pseudo-Boolean functions. *Studii si Cercetari Matematice*, *14*, 359–364. (in Romanian)

Ishikawa, H. (2011, June). Transformation of general binary MRF minimization to the first-order case. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, *33*(6), 1234–1249.

Kolmogorov, V., & Zabih, R. (2004, Feb). What energy functions can be minimized via graph cuts? *IEEE Transactions on Pattern Analysis and Machine Intelligence*, *26*(2), 147–159.