Persistency of Linear Programming Relaxations for the Stable Set Problem

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- Can be modeled as an Integer Programming problem

$$\begin{aligned} \max \quad & \sum_{v \in V(G)} w_v x_v \\ \text{s. t.} \quad & x_v + x_w \leq 1 & \forall \{v, w\} \in E(G) \\ & x_v \in \{0, 1\} & \forall v \in V(G) \end{aligned}$$

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- The *edge relaxation* $R_{\text{edge}}(G)$ is the set of feasible points of the LP-relaxation of the model above.

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- For example, to apply an LP-based branch-and-bound type of algorithm.

Persistency

Definition: Persistency

A polytope $P \subseteq [0,1]^n$ has the *persistency property* if for every objective vector $c \in \mathbb{R}^n$ and every c-maximal point $x \in P$, there exists a c-maximal integer point $y \in P \cap \{0,1\}^n$ such that $x_i = y_i$ for each $i \in \{1,2,\dots,n\}$ with $x_i \in \{0,1\}$.

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  then there exists an optimal stable set $S^\star$ containing every node in V_1 and not containing any node in V_0.
- The variables corresponding to nodes in $V_0 \cup V_1$ can be deleted.
- We can reduce the dimension of our problem.

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  - UQBP persistency has been used in the computer vision community to solve very large image restoration problems (Fix et al., 2015; Ishikawa, 2011; Kolmogorov & Rother, 2007).
  - There exists a polynomial-time algorithm to compute the largest possible sets V_0 and V_1 for UQBP (Boros et al., 2008).

Formulations

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Definition: Formulation

Let $\mathcal{G}$ be the set of finite undirected simple graphs.

We regard an LP *formulation* for the stable set problem as a map that assigns to every graph $G \in \mathcal{G}$ a polytope $R(G)$ such that $R(G) \cap \mathbb{Z}^{V(G)} = P_{\text{conv}}(G) \cap \mathbb{Z}^{V(G)}$.

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  - clique inequalities,
  - odd-cycle inequalities, ...

Question:

Do there exist stronger linear programming formulations for the stable set problem that also have the persistency property for every graph G?

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Main result:

- Negative answer, for formulations satisfying certain mild conditions.

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Theorem (Rodríguez-Heck, Stickler, Walter, Weltge (2019))

Let R be a formulation satisfying *some mild conditions*. Then $R(G)$ has the persistency property for all graphs $G \in \mathcal{G}$ if and only if $R \equiv R_{\text{edge}}$ or $R \equiv P_{\text{conv}}$.

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- Two formulations R^1 and R^2 are *equivalent* ($R^1 \equiv R^2$) if $R^1(G) = R^2(G)$ holds for every $G \in \mathcal{G}$.

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(C) For every pair of graphs $G_1, G_2 \in \mathcal{G}$, and $v_1 \in G_1$, $v_2 \in G_2$,

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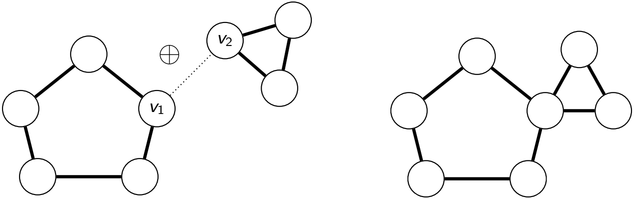
Definition: 1-sum of polytopes $R(G_1) \oplus_{v_2}^{v_1} R(G_2)$

$\text{conv}(\{(x, y) \in R(G_1) \times R(G_2)\} \mid x_{v_1} = y_{v_2})$

1-sum of graphs $G_1 \oplus_{v_2}^{v_1} G_2$



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Some well-known formulations

Corollary

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$$R_{\mathsf{clq}}(G) = \{x \in R_{\mathsf{edge}}(G) \mid x(V(C)) \leq 1 \text{ for each clique } C \text{ of } G\}$$

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do not have the persistency property for all graphs $G \in \mathcal{G}$

Proof idea

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Let R be a formulation satisfying *some mild conditions*. Then $R(G)$ has the persistency property for all graphs $G \in \mathcal{G}$ if and only if $R \equiv R_{\text{edge}}$ or $R \equiv P_{\text{conv}}$.

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- "only if" part restated

If there exist graphs $G_1, G_2 \in \mathcal{G}$ with $R(G_1) \neq R_{\text{edge}}(G_1)$ and $R(G_2) \neq P_{\text{conv}}(G_2)$, then there exists a graph $G^\star$ for which the polytope $R(G^\star)$ does not have the persistency property.

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- ... what is the challenge?
  - R can be "anything"!
  - For example, R can be defined by all facets of P_{conv} except one facet, or two facets, or ...

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Proof idea

If there exist graphs $G_1, G_2 \in \mathcal{G}$ with $R(G_1) \neq R_{\text{edge}}(G_1)$ and $R(G_2) \neq P_{\text{conv}}(G_2)$, then there exists a graph $G^\star$ for which the polytope $R(G^\star)$ does not have the persistency property.

- We "only" have to construct a counter-example...
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  - Fortunately, R satisfies our *mild conditions*.

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Constructing the counterexample $G^\star$

Lemma

There exist: a graph G, a vector $c \in \mathbb{R}^{V(G)}$ and a node $v \in V(G)$ such that

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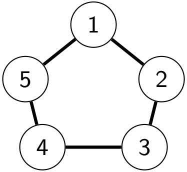

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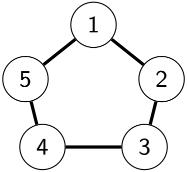

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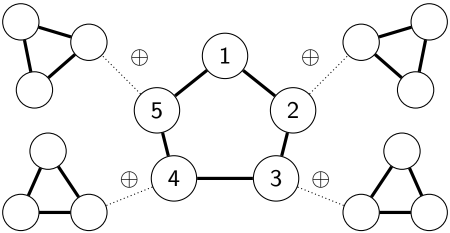

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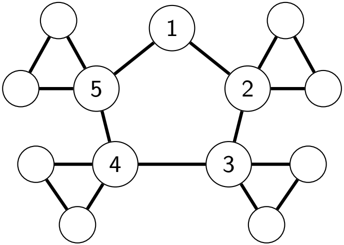

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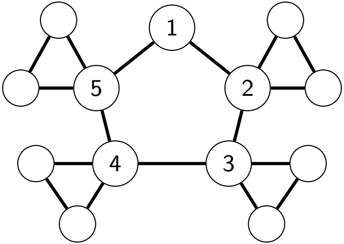

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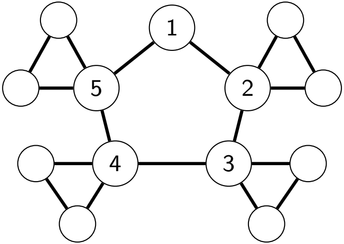

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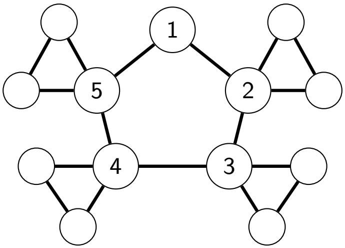

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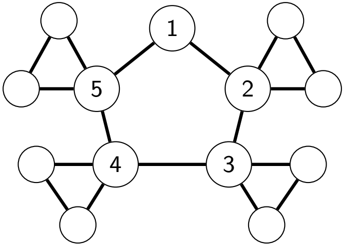

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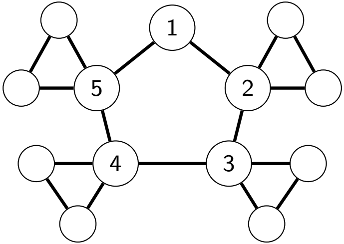

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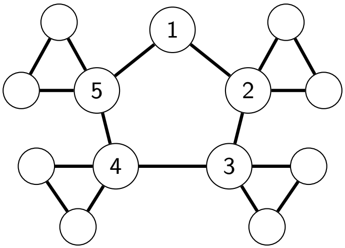

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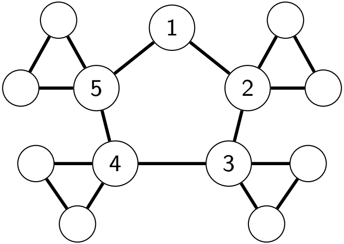

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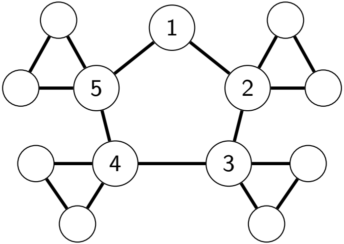

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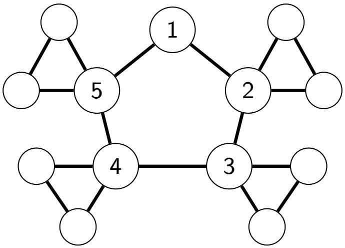

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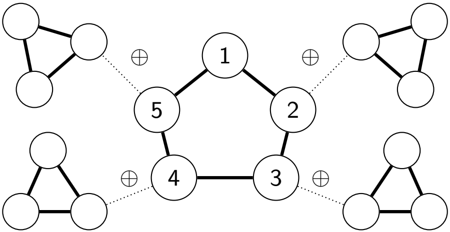

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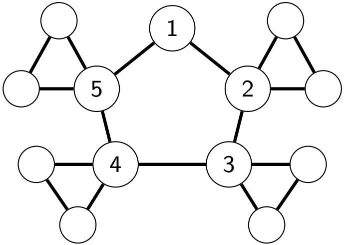

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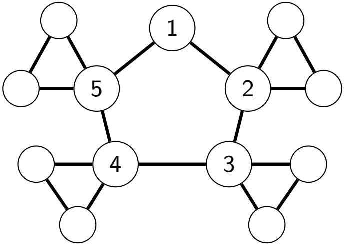

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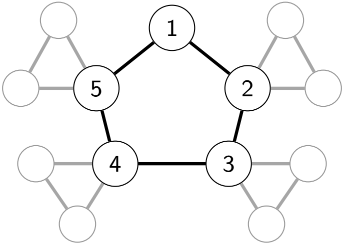

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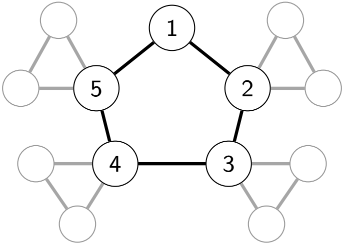

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Let $\sum_{j \in V(G)} a_j x_j \leq b$ be a facet-defining inequality for $P_{\text{conv}}(G)$ that is neither a bound nor an edge inequality. Then

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Sewell's result

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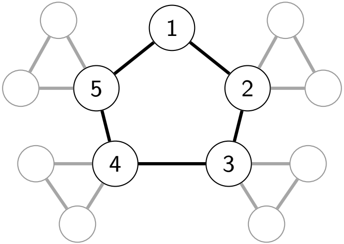

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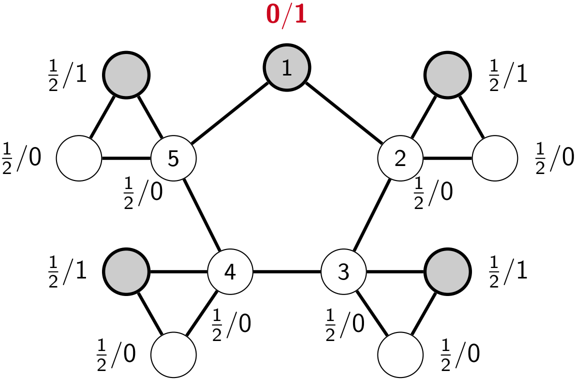
$$a_1 \leq \sum_{j \in V(G)} a_j - 2b \Rightarrow b \leq \sum_{j \in V(G) \setminus \{1\}} a_j \cdot \frac{1}{2}$$

Contradiction of persistency for Example 1



- $G^\star$ and LP/IP maxima.
- $c^\star$ takes value $\varepsilon = \frac{1}{20}$ for node 1, and value 1 for every other node.

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- Solution $\hat{x}$ has to satisfy all *unknown* facets of $R(G^\star)$!

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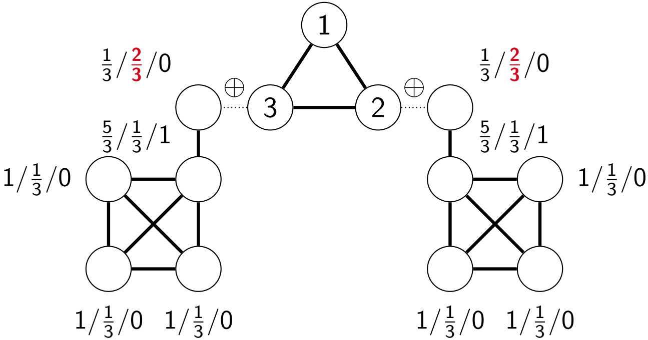
Example 2: $R = R^{\text{oc3}}_{\text{edge}}$. Separately: $c^\star$/LP/IP maxima.



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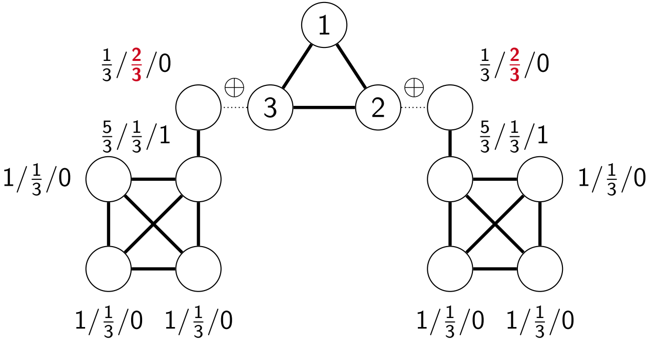
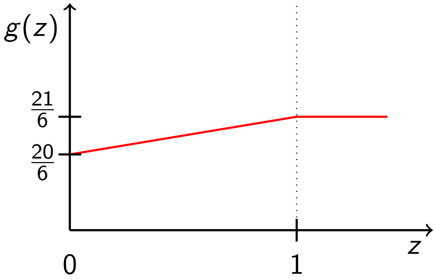


$$g(z) = \max\Big\{ \sum_{j \in \{2,3\}} f(x_j) \mid x_1 + x_2 + x_3 \le z,\ x \in R_{\text{edge}}(C_3) \Big\}$$

$$f(x_j) = \max\Big\{ \frac{1}{3} x_{A'^j} + \frac{5}{3} x_{A^j} + x_{B^j} + x_{C^j} + x_{D^j} x \in R^{\text{oc3}}_{\text{edge}}(G^j) \text{ and } x_{A'^j} = x_j \Big\}$$

Hidden difficulties

Example 2: $R = R_{\text{edge}}^{\text{oc3}}$. Function g is strictly increasing in $[0, b] = [0, 1]$, x_1 does not contribute to maximum.



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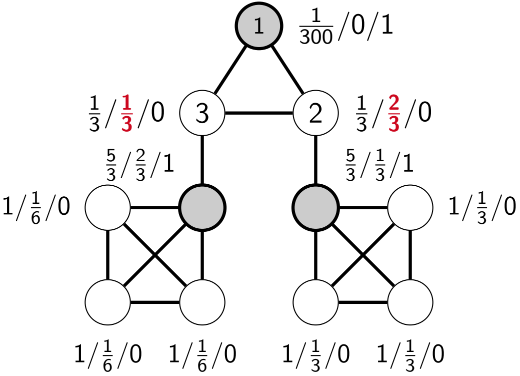
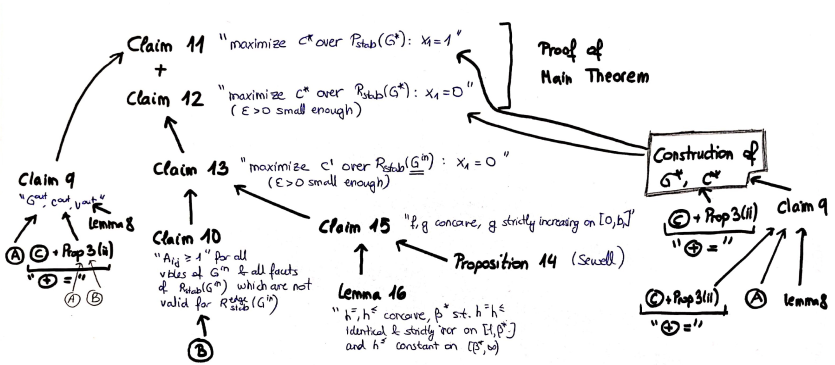
Example 2: $R = R^{\text{oc3}}_{\text{edge}}$. For small ε, $x_1 = 0$ in all maximizers.



Other hidden difficulties...



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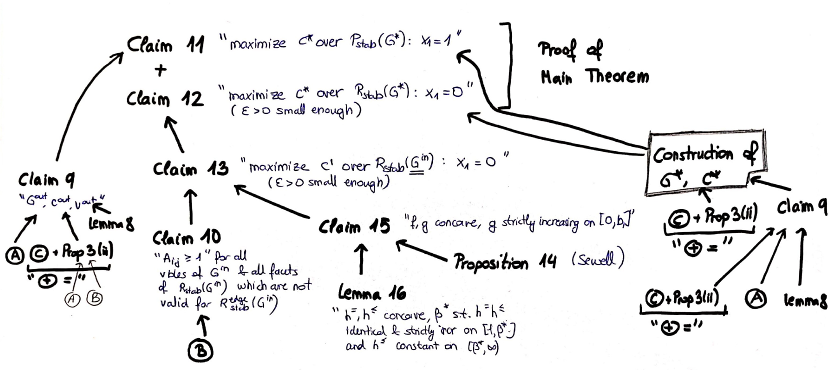

Thank you for your attention!

RWTH AACHEN UNIVERSITY

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Persistency of the edge relaxation

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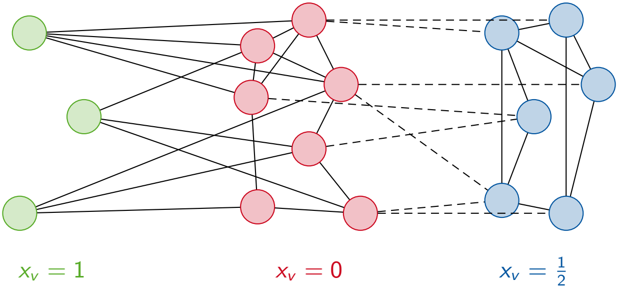
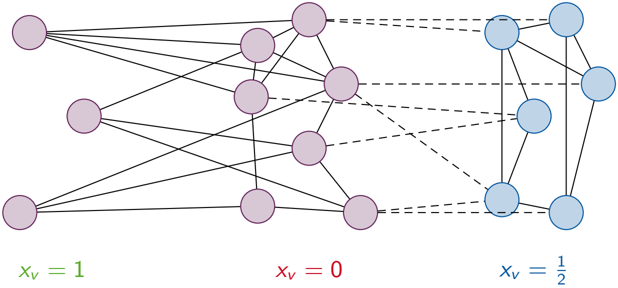


Persistency of the edge relaxation

- If $V_1 = \{v \in V \mid x_v = 1\}$ is c-maximal in the subgraph $G[U]$ ($U = V_1 \cup V_0$) then there exists a c-maximal stable set in G containing V_1.



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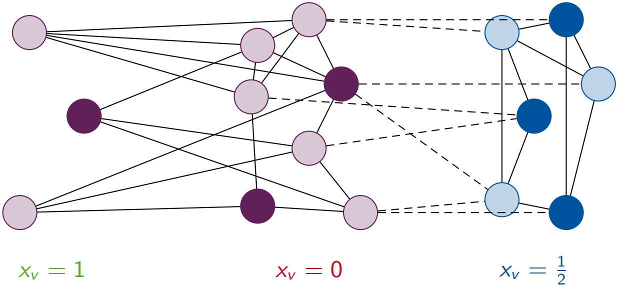

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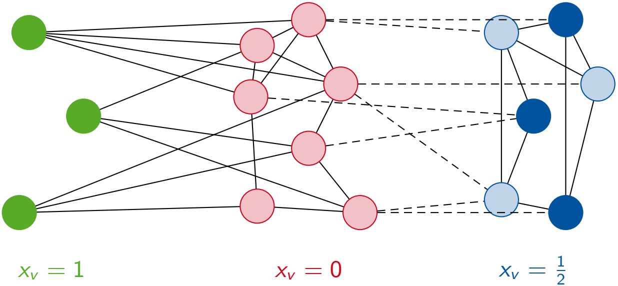

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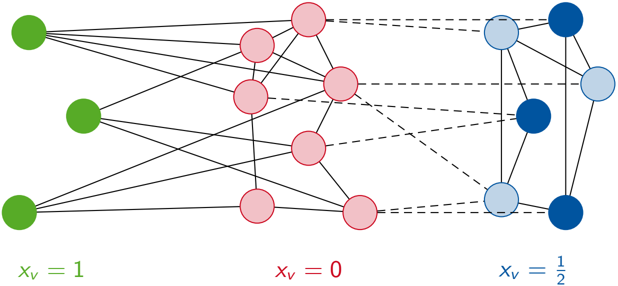

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- It can be seen that V_1 is c-maximal in $G[U]$: see (Nemhauser & Trotter, 1975).

What are the *mild conditions* (more in detail)?

(A) The inequalities defining R are derived from facets of P_{conv}.

(B) Validity of facet-defining inequalities of $R(G)$ is inherited by induced subgraphs.

(C) For every pair of graphs $G_1, G_2 \in \mathcal{G}$, and $v_1 \in G_1$, $v_2 \in G_2$,

$$R(G_1 \oplus_{v_2}^{v_1} G_2) = R(G_1) \oplus_{v_2}^{v_1} R(G_2)$$

Definition: 1-sum of graphs $G_1 \oplus_{v_2}^{v_1} G_2$

The graph obtained from the disjoint union of G_1 and G_2 by identifying v_1 with v_2.

Definition: 1-sum of polytopes $R(G_1) \oplus_{v_2}^{v_1} R(G_2)$

$\operatorname{conv}(\{(x, y) \in R(G_1) \times R(G_2)\} \mid x_{v_1} = y_{v_2})$

What are the *mild conditions* (more in detail)?

(A) For each $G \in \mathcal{G}$, each inequality with support $U \subseteq V(G)$ that is facet-defining for $R(G)$ is also facet-defining for $P_{\text{conv}}(G[U])$

(B) For each $G \in \mathcal{G}$, each inequality with support $U \subseteq V(G)$ that is facet-defining for $R(G)$ is valid for $R(G[U])$.

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