Extended Formulations for Radial Cones of Odd-Cut Polyhedra

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Joint work with

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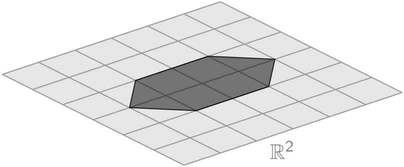
Aussois Combinatorial Optimization Workshop 2019

Operations Research | RWTH AACHEN UNIVERSITY

Extended formulations:

- Consider a polyhedron P of interest.



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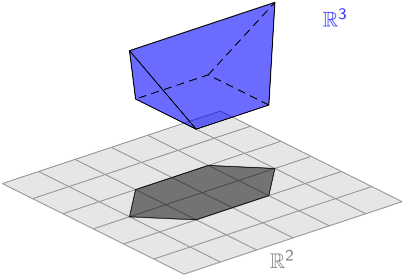

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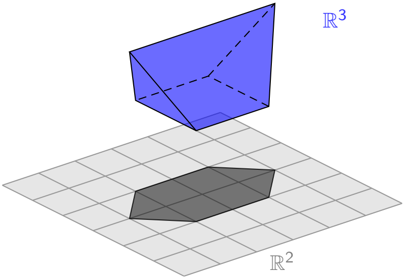

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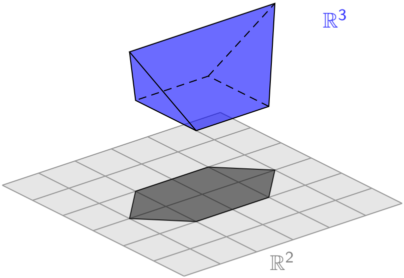

Matchings:

- A perfect matching in a graph $G = (V, E)$ is a set $M \subseteq E$ with $|M \cap \delta(v)| = 1$.
- The weighted perfect matching problem can be solved in polynomial time (Edmonds, 1965).

Theorem (Rothvoss, 2013)

For every even n, the extension complexity of the perfect-matching polytope for K_n is at least $2^{\Omega(n)}$.

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- Improve a given feasible solution of a combinatorial optimization problem or determine optimality.

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Augmentation & Radial Cones

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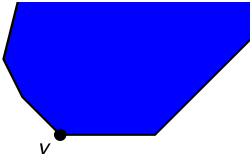
Polyhedral version:

- Consider $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, objective vector $c \in \mathbb{R}^n$, and point $v \in P$.
- Determine optimality or find improving direction $d \in \mathbb{R}^n$ with $v + d \in P$.



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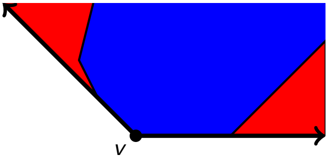
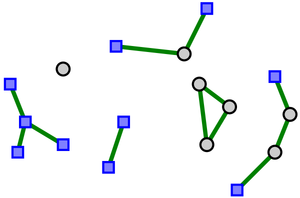
- Polyhedron for this task is radial cone:

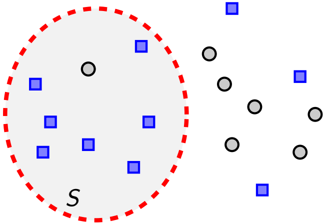


$$K_P(v) := \text{cone}(P - v) + v = \{x \in \mathbb{R}^n : A_{i,*}x \leq b_i \text{ for all } i \text{ with } A_{*,i}v = b_i\}$$

Definitions ($K_n = (V_n, E_n)$ complete graph on n nodes; $T \subseteq V$, $|T|$ even)**:**

- $J \subseteq E$ is a T-join if $|J \cap \delta(v)|$ is odd $\iff v \in T$
- $C = \delta(S) \subseteq E$ is a T-cut if $|S \cap T|$ is odd.





T-Joins & T-Cuts

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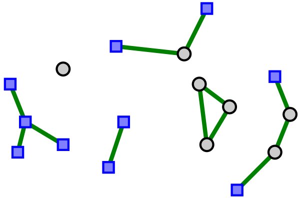

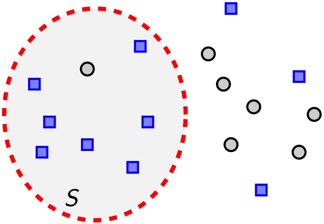

Facts:

- Both minimization problems can be solved in polynomial time for $c \geq \mathbb{O}$.
- Each T-join J intersects each T-cut C in at least one edge:

$$|J \cap C| = \langle \chi(J), \chi(C) \rangle \geq 1$$

T-Join- and T-Cut-Polyhedra

Polyhedra (Edmonds & Johnson, 1973)**:**

- T-join Polyhedron $P_{T\text{-join}}(n)^{\uparrow}$:

$$\begin{aligned} \langle \chi(C), x \rangle &\geq 1 \quad \text{for each } T\text{-cut } C \\ x_e &\geq 0 \quad \text{for each } e \in E \end{aligned}$$

- T-cut Polyhedron $P_{T\text{-cut}}(n)^{\uparrow}$:

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- Focus on the case $T = V$.
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Theorem (Ventura & Eisenbrand, 2003)

For even n and every vertex v of $P_{V\text{-join}}(n)^{\uparrow}$, corresponding to a V-join $J \subseteq E_n$ in K_n, the extension complexity of the radial cone of $P_{V\text{-join}}(n)$ at v is at most $\mathcal{O}(|J| \cdot n^2)$.

Result Preview

From their paper:

4.3. Open problems

This compact formulation for the active cone of a given perfect matching could be given, since the parity condition of the tight cuts can be ensured by considering each crossing edge individually. A direction of future research could be to find out, whether this primal view can be helpful to find compact linear formulations of active cones of polyhedra for other classes of combinatorial problems. Interesting candidates might be the stable-set polyhedron of a claw-free graph or the odd-cut polyhedron, which is the blocker of the T-join polyhedron.

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Hard work by the night:

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Theorem (Walter & Weltge, 2018)

For even n and vertices v of $P_{V\text{-cut}}(n)^{\uparrow}$, the extension complexity of the radial cone of $P_{V\text{-cut}}(n)$ at v is at least $2^{\Omega(n)}$.

Blocking Polarity: Basics

Definitions:

- A polyhedron $P \subseteq \mathbb{R}^d_+$ is blocking if $x' \geq x$ implies $x' \in P$ for all $x \in P$.
- Possible descriptions are:

$$P = \{x \in \mathbb{R}^d_+ : \langle y^{(i)}, x \rangle \geq 1 \text{ for } i = 1, \ldots, m\} \qquad (y^{(1)}, \ldots, y^{(m)} \in \mathbb{R}^d_+)$$

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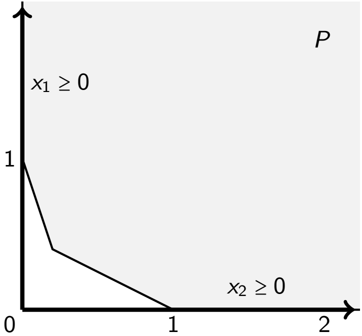

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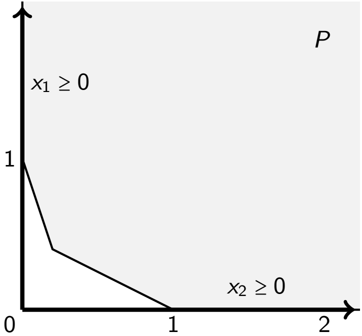

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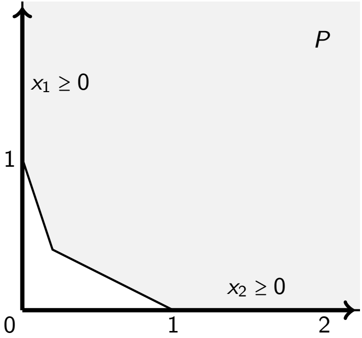

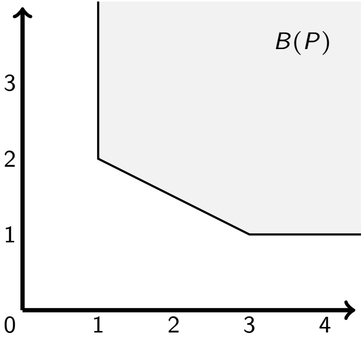

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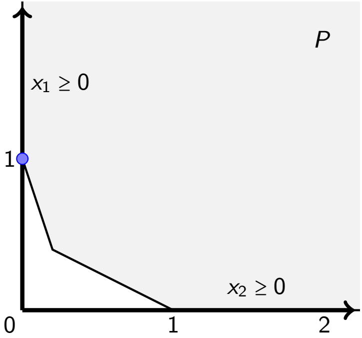

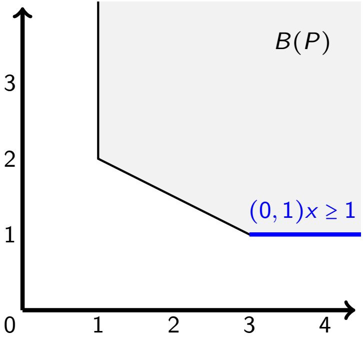

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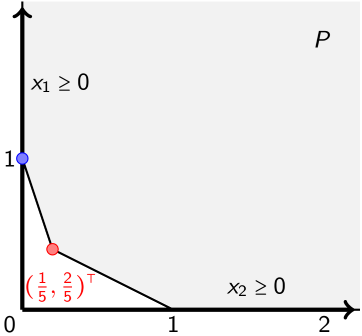

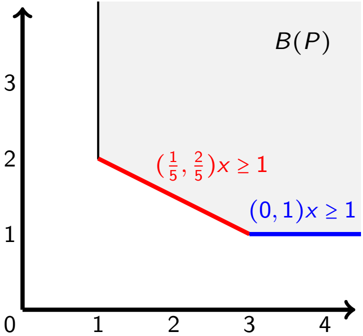

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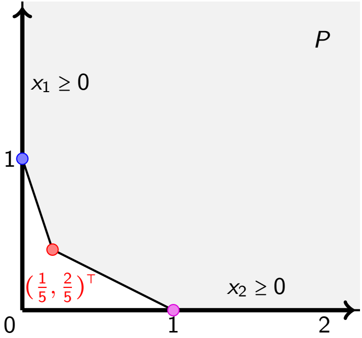

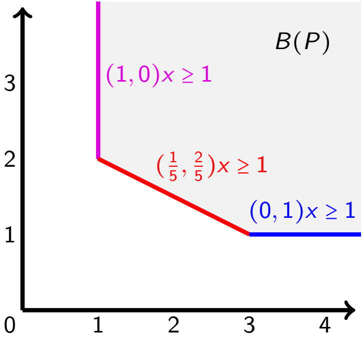

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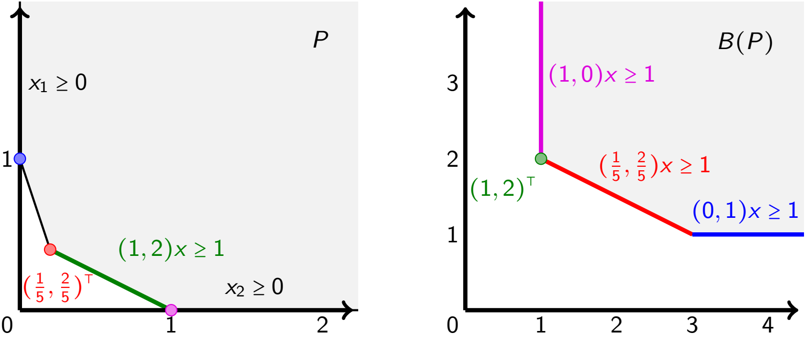

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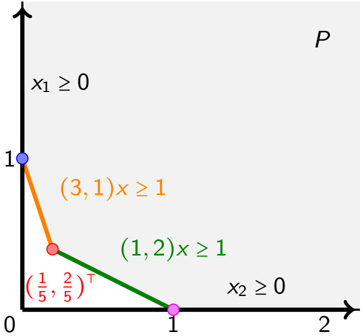

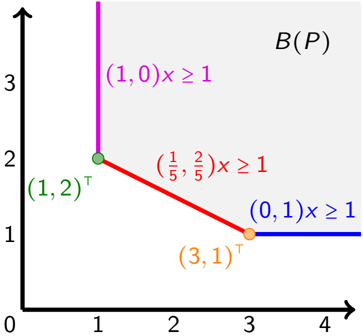

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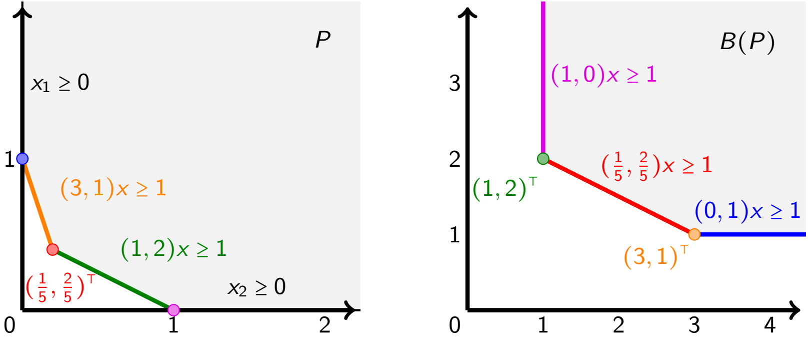
- If P is blocking, then $B(B(P)) = P$.



Blocking Polarity: Extensions

Proposition (Martin, 1991; Conforti, Kaibel, Walter & Weltge, 2015)

Given a non-empty polyhedron Q and $\gamma \in \mathbb{R}$, let

$$P := \{x : \langle y, x \rangle \geq \gamma \text{ for all } y \in Q\}.$$

Then $\text{xc}(P) \leq \text{xc}(Q) + 1$.

Proof:

- Let $Q = \{Tz : Az \leq b\}$, where A has $m = \text{xc}(Q)$ rows.

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$$\hat{x} \in P \iff \min\{\langle \hat{x}, y\rangle : y \in Q\} \geq \gamma$$

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- Let $Q = \{Tz : Az \leq b\}$, where A has $m = \mathrm{xc}(Q)$ rows.

$$\begin{aligned} \hat{x} \in P &\iff \min\{\langle \hat{x}, y \rangle : y \in Q\} \geq \gamma \\ &\iff \min\{\langle \hat{x}, Tz \rangle : Az \leq b\} \geq \gamma \end{aligned}$$

Blocking Polarity: Extensions

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Consequences:

- P and $B(P)$ have (essentially) the same extension complexity.
- $2^{\Omega(|T|)} \leq \mathrm{xc}(P_{T\text{-cut}}(n)^\uparrow)$.

Polar object of radial cone:

- Any $v \in P$ defines a face $F_{B(P)}(v) := \{y \in B(P) : \langle v, y \rangle = 1\}$ of $B(P)$.

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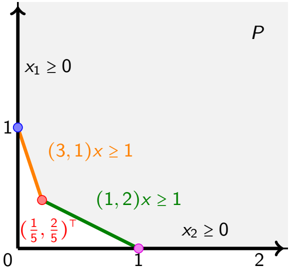

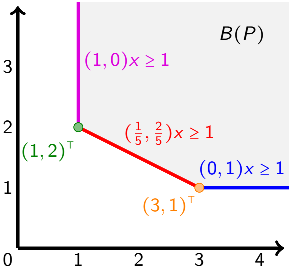

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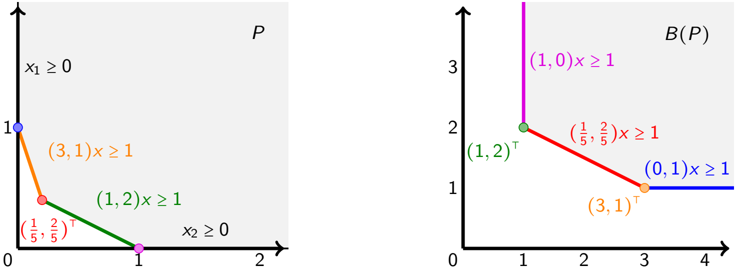

Consequence:

- To prove lower or upper bounds on $\mathrm{xc}(K_P(v))$, analyze $F_{B(P)}(v)$!

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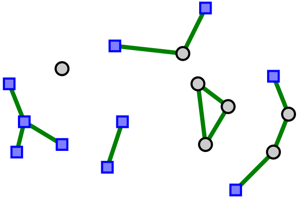

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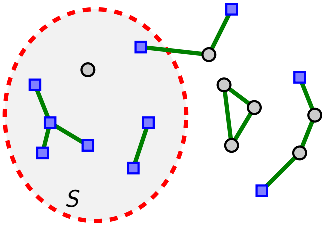

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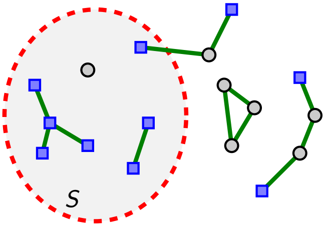

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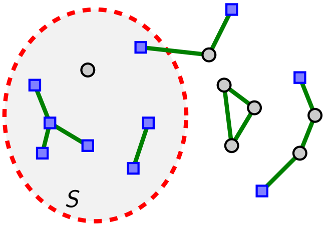

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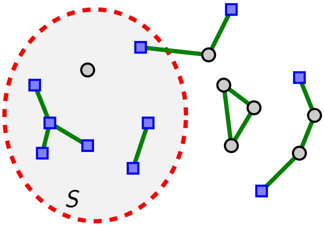

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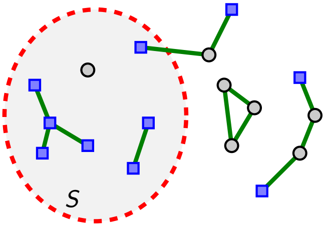
- P is convex hull of union of all F_m and disjunctive programming yields the result.



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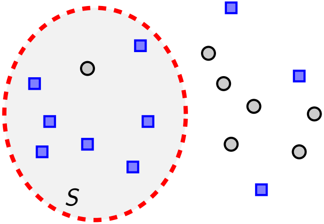

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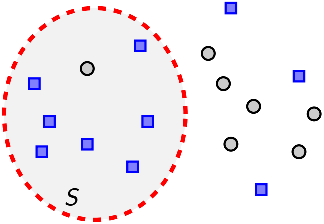

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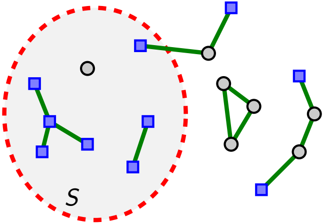

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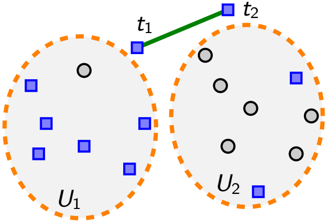

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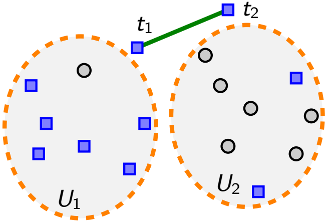

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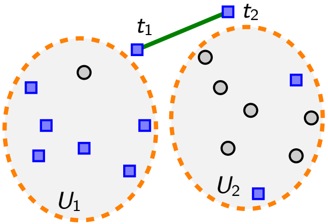

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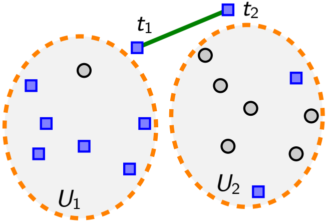

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Other candidates for investigation:

- Stable-set polytopes of claw-free graphs (maybe next year ...)
- Stable-set polytopes of perfect graphs (polyhedral description is known, but best (known) extended formulation has $\mathcal{O}(n^{\log n})$ facets).