Extended Formulations for Radial Cones

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Joint work with

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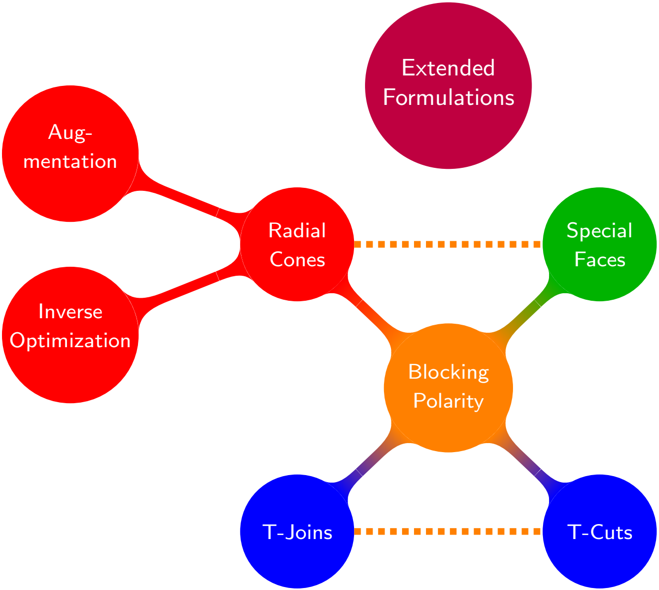
Operations Research

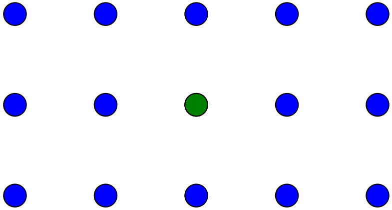
RWTH AACHEN UNIVERSITY

Overview


Extended Formulations
Aug-mentation
Radial Cones
Special Faces
Inverse Optimization
Blocking Polarity
T-Joins
T-Cuts

Polyhedral method:

- Consider feasible solutions $\mathcal{F} \subseteq 2^E$ over some ground set E and an objective vector $c \in \mathbb{R}^E$ with the goal of minimizing $c(F) := \sum_{e \in F} c_e$.
- Identify $F \in \mathcal{F}$ with $\chi(F) \in \{0,1\}^E$ s.t. $\chi(F)_e = 1 \iff e \in F$.



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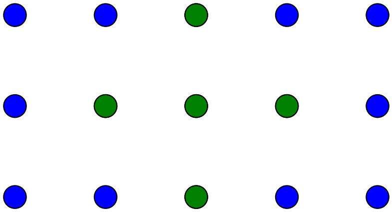
- Optimization problem is then to minimize $\langle c, x \rangle$ over $x \in X$.



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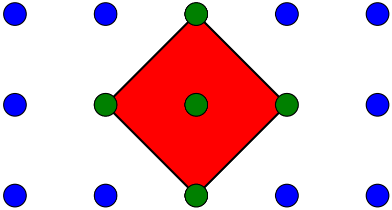

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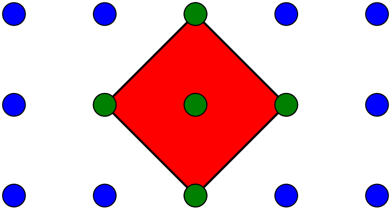
- Let $X := \{\chi(F) : F \in \mathcal{F}\} \subseteq \{0,1\}^E$.
- Optimization problem is then to minimize $\langle c, x \rangle$ over $x \in \text{conv}(X)$.
- Find an outer description of $\text{conv}(X)$, i.e., $\text{conv}(X) = \left\{x \in \mathbb{R}^E : Ax \leq b\right\}$.
- Optimization problem is now an LP and we can use black-box solvers.[1]



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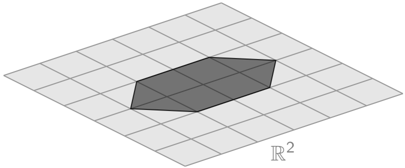
- $P = \text{conv}(X)$ has many facets, but maybe there exists an extension (Q, π) ($Q \subseteq \mathbb{R}^d$, $\pi : \mathbb{R}^d \to \mathbb{R}^n$ linear with $P = \pi(Q)$) with few facets?



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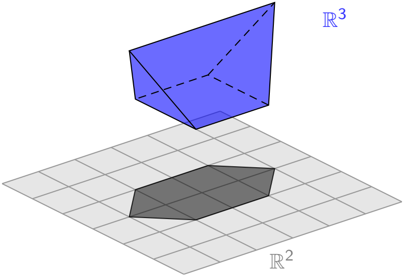

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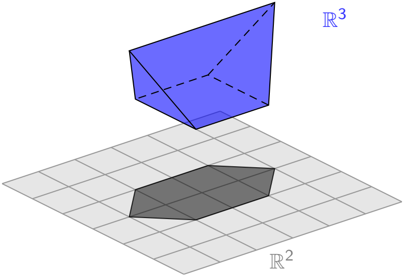

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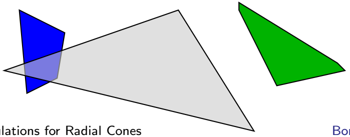

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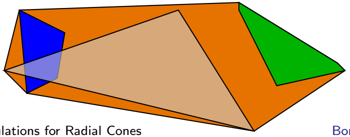

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- Applying the theorem: $\text{xc}(\text{conv}(X)) \leq \mathcal{O}(n^2)$

Limitations

Hard problems:

- Max-Cut problem: cut polytope for K_n (complete graph with n nodes) has extension complexity $2^{\Omega(n)}$ (Fiorini, Massar, Pokutta, Tiwary & de Wolf, 2012), best bound is 1.5^n (Kaibel & Weltge, 2013).

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Theorem (Rothvoss, 2013)

For every even n, $\text{xc}(P_{pmatch}(n)) \geq 2^{\Omega(n)}$. Here, $P_{pmatch}(n)$ denotes the perfect matching polytope for K_n.

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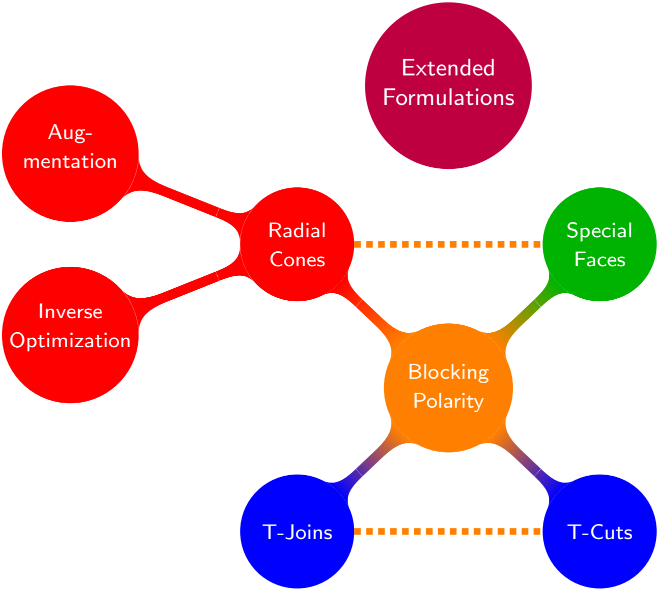
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Idea:

- Suppose $c \in \{0,1\}^E$, how many augmentation steps will you need?
- Apply bit scaling.

Polyhedral version of the augmentation problem:

- Consider a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ and an objective vector $c \in \mathbb{R}^n$.
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- The polyhedron for this task is the radial cone:

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K_P(v) &:= \operatorname{cone}(P - v) + v \\
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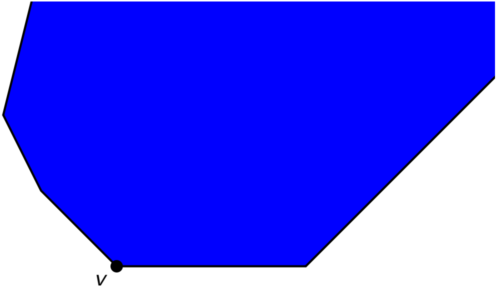

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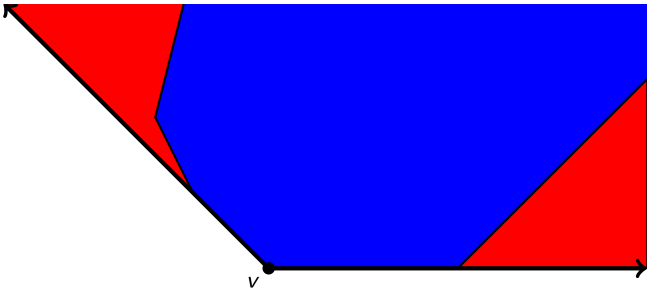

Inverse problem:

- Input: $\hat{F} \in \mathcal{F}$ and $\hat{c} \in \mathbb{R}^E$
- Goal: minimize $\|c - \hat{c}\|$ over $c \in \mathbb{R}^E$ such that $\hat{F}$ maximizes c.
- Application: find objective for observed behavior $\hat{F}$ that is assumed to be optimal.

Inverse Optimization Problem

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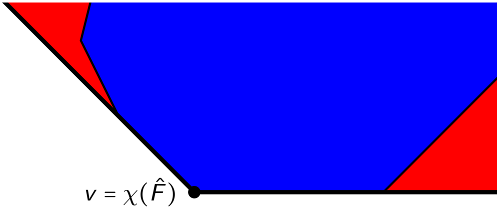
- Goal: minimize $||c - \hat{c}||$ over $c \in \mathbb{R}^E$ such that $\hat{F}$ maximizes c.
- Application: find objective for observed behavior $\hat{F}$ that is assumed to be optimal.



Inverse Optimization Problem

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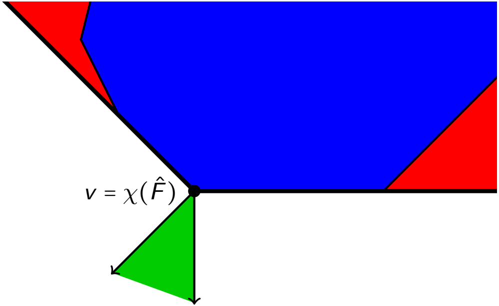

Feasible solutions of inverse optimization problem:

- Set of feasible c-vectors is the polar cone of cone$(P - v)$.

Radial Cones: Basic Results

Nice problems:

- For $v \in P$ we have $\text{xc}(K_P(v)) \leq \text{xc}(P)$.
- Consequence: nice polyhedra have nice radial cones.

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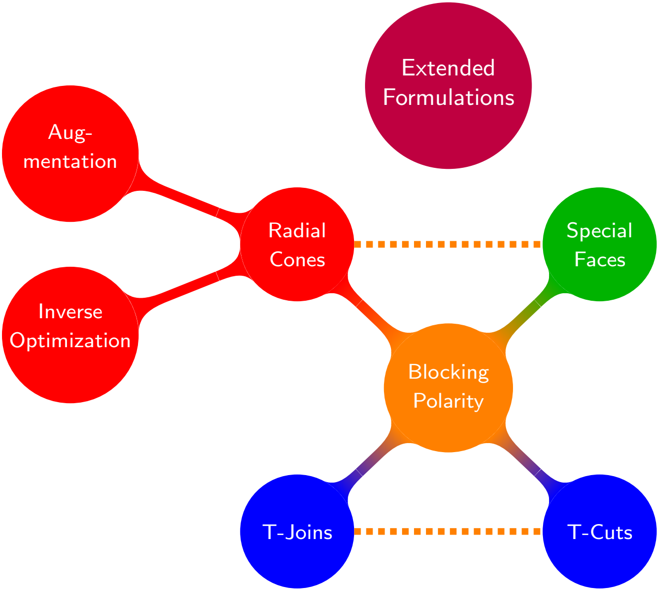
Which polyhedra remain?

- Matching polytopes & friends (this talk)
- Stable-set polytopes of claw-free or perfect graphs
- Beat known bounds for nice polyhedra

Overview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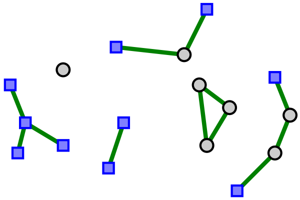
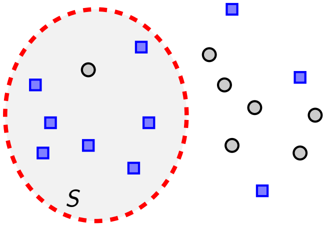
T-Joins & T-Cuts

Definitions ($K_n = (V_n, E_n)$ complete graph on n nodes; $T \subseteq V$, $|T|$ even)**:**

- $J \subseteq E$ is a T-join if $|J \cap \delta(v)|$ is odd $\iff v \in T$



- $C = \delta(S) \subseteq E$ is a T-cut if $|S \cap T|$ is odd.



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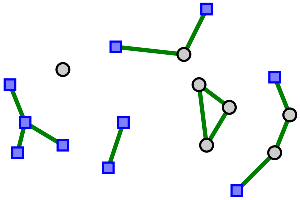

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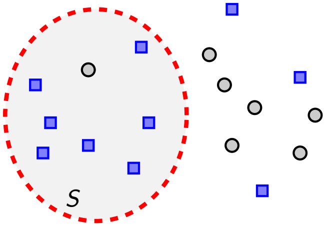

Facts:

- Both minimization problems can be solved in polynomial time for $c \geq \mathbb{O}$.
- Each T-join J intersects each T-cut C in at least one edge:

$$|J \cap C| = \langle \chi(J), \chi(C) \rangle \geq 1$$

T-Join- and T-Cut-Polyhedra

Polyhedra (Edmonds & Johnson, 1973)**:**

- T-join Polyhedron $P_{T\text{-join}}(n)^{\uparrow}$:

$$\begin{aligned} \langle \chi(C), x \rangle &\geq 1 \quad \text{for each } T\text{-cut } C \\ x_e &\geq 0 \quad \text{for each } e \in E \end{aligned}$$

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- A T-join $J \subseteq E$ is a perfect matching on nodes T if and only if $x = \chi(J)$ satisfies the valid inequalities $x_e \geq 0$ for all $e \in E \setminus E[T]$ and $\sum_{e \in \delta(v)} x_e \geq 1$ for all $v \in T$ with equality.

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Proposition (Walter & Weltge, 2018)

For every n and every set $T \subseteq V_n$, $\text{xc}(P_{T\text{-join}}(n)^{\uparrow}) \leq \mathcal{O}(n^2 \cdot 2^{|T|})$.

Idea:

- For each $S \subseteq T$ with $|S| = \frac{1}{2}|T|$, consider the b-flow polyhedron for $b_v = -1$ for all $v \in S$, $b_v = 1$ for all $v \in T \setminus S$ and $b_v = 0$ otherwise.
- Apply disjunctive programming over all such polyhedra.

Theorem (Ventura & Eisenbrand, 2003)

For every set $T \subseteq V_n$ with $|T|$ even and every vertex v of $P_{T\text{-join}}(n)^{\uparrow}$, corresponding to a T-join $J \subseteq E_n$ in K_n, the extension complexity of the radial cone of $P_{T\text{-join}}(n)$ at v is most $\mathcal{O}(|J| \cdot n^2)$.

Their proof: ad-hoc construction using sets of flow variables.
Our new proof: via blocking polarity

Result Preview

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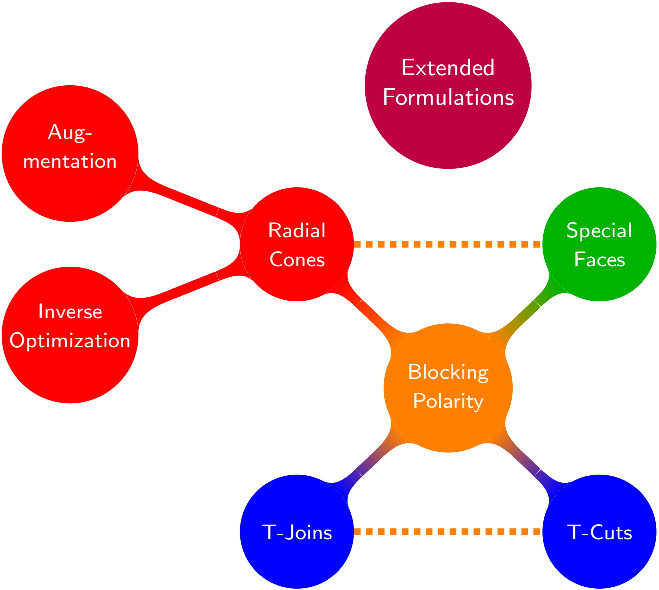
Theorem (Walter & Weltge, 2018)

For sets $T \subseteq V_n$ with $|T|$ even and vertices v of $P_{T\text{-cut}}(n)^{\uparrow}$, the extension complexity of the radial cone of $P_{T\text{-cut}}(n)$ at v is least $2^{\Omega(|T|)}$.

Our proof: via blocking polarity

Overview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Blocking Polarity: Basics

Definitions:

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$$P = \{x \in \mathbb{R}^d_+ : \langle y^{(i)}, x \rangle \geq 1 \text{ for } i = 1, \dots, m\} \qquad (y^{(1)}, \dots, y^{(m)} \in \mathbb{R}^d_+)$$

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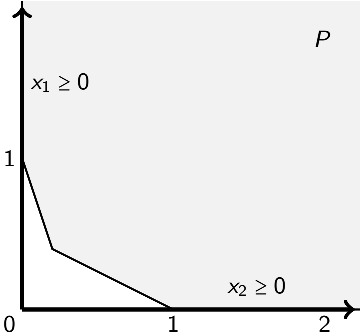

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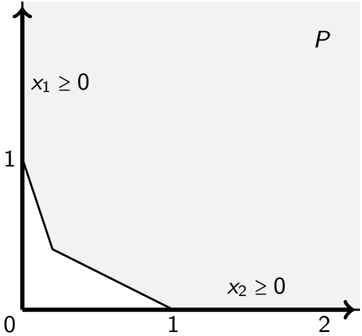

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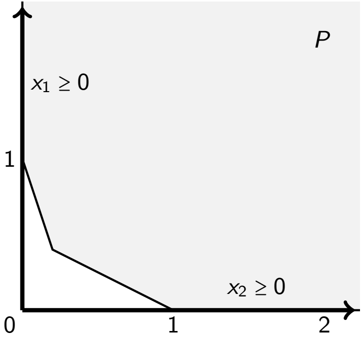

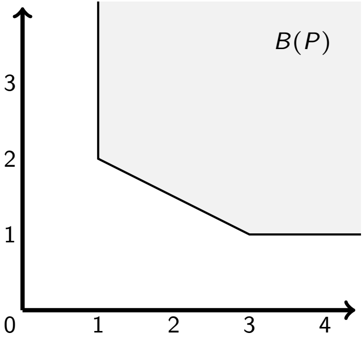

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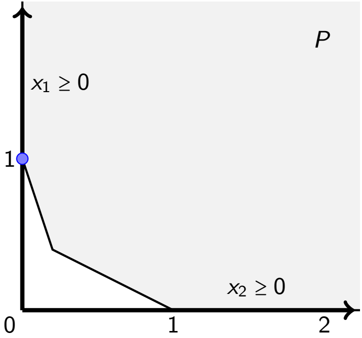

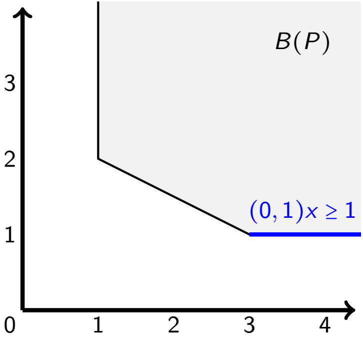

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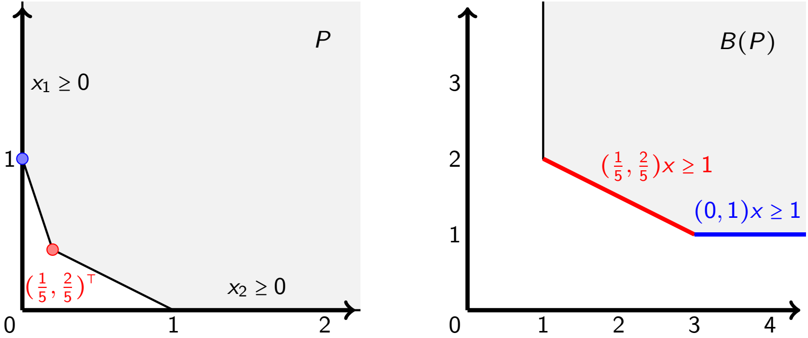

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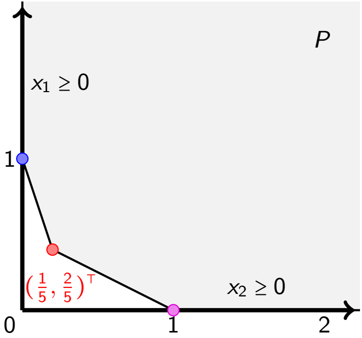

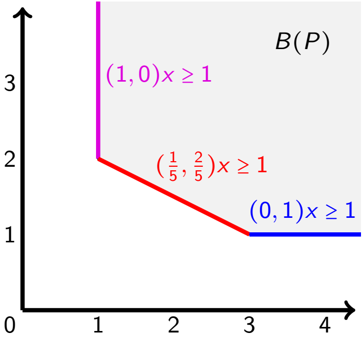

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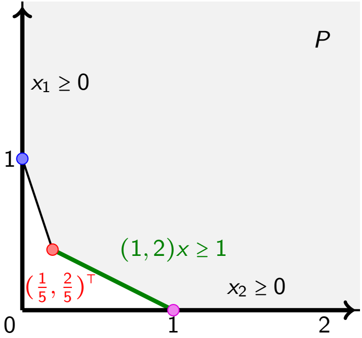

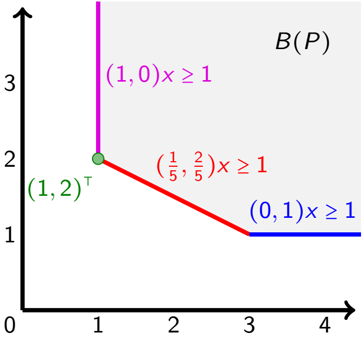

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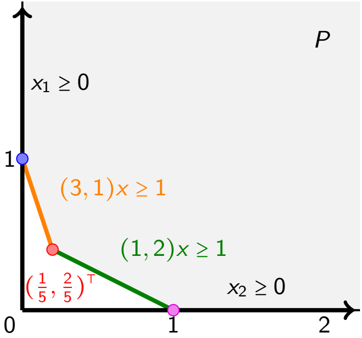

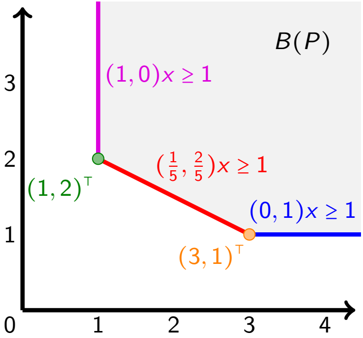

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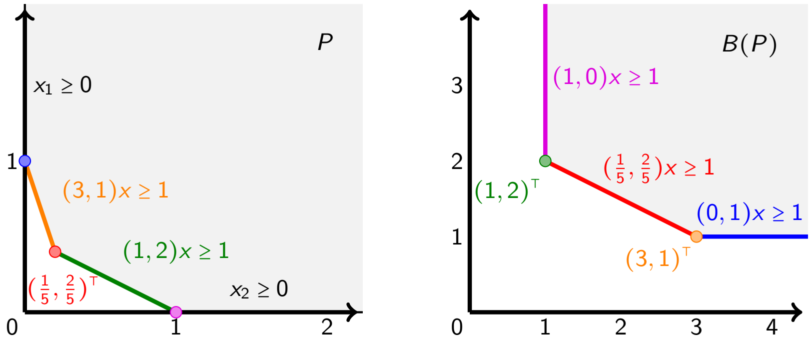

Blocking Polarity: Extensions

Proposition (Martin, 1991; Conforti, Kaibel, Walter & Weltge, 2015)

Given a non-empty polyhedron Q and $\gamma \in \mathbb{R}$, let

$$P := \{x : \langle y, x \rangle \geq \gamma \text{ for all } y \in Q\}.$$

Then $\text{xc}(P) \leq \text{xc}(Q) + 1$.

Proof:

- Let $Q = \{Tz : Az \leq b\}$, where A has $m = \text{xc}(Q)$ rows.

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$$\hat{x} \in P \iff \min\{\langle \hat{x}, y \rangle : y \in Q\} \geq \gamma$$

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Consequences:

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Blocking Polarity: Radial Cones

Polar object of radial cone:

- Any $v \in P$ defines a face $F_{B(P)}(v) := \{y \in B(P) : \langle v, y \rangle = 1\}$ of $B(P)$.

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Lemma

Let $P \subseteq \mathbb{R}^d_+$ be a blocking polyhedron and let $v \in P$.

(i) $F_{B(P)}(v) = \left\{y \in \mathbb{R}^d : \langle v, y \rangle = 1,\ \langle x, y \rangle \geq 1 \ \forall x \in K_P(v)\right\}$.

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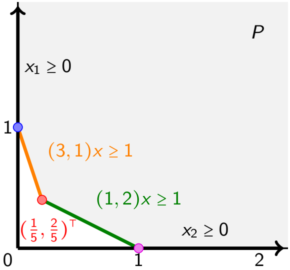

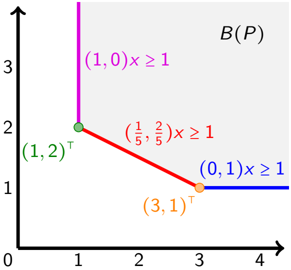

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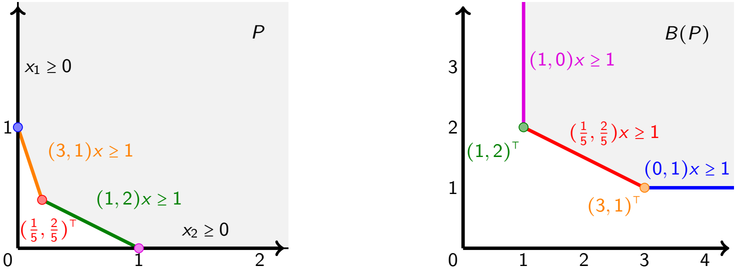

Consequence:

- $\mathrm{xc}(K_P(v))$ and $\mathrm{xc}(F_{B(P)}(v))$ differ by at most 1.
- To prove lower or upper bounds on $\mathrm{xc}(K_P(v))$, analyze $F_{B(P)}(v)$!

Radial Cones of T-Join Polyhedra

Theorem (Ventura & Eisenbrand, 2003)

For every set $T \subseteq V_n$ with $|T|$ even and every vertex v of $P_{T\text{-join}}(n)^{\uparrow}$, corresponding to a T-join $J \subseteq E_n$ in K_n, the extension complexity of the radial cone of $P_{T\text{-join}}(n)$ at v is most $\mathcal{O}(|J| \cdot n^2)$.

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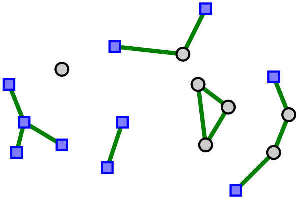

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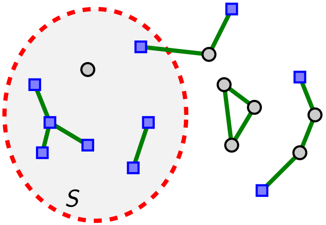

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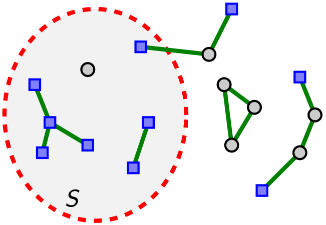

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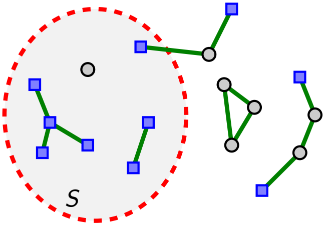

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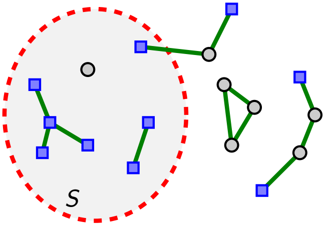

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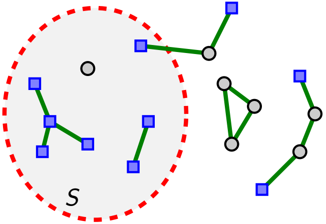

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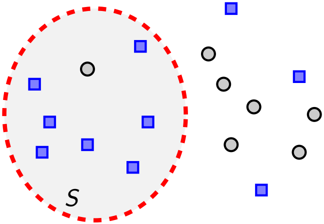

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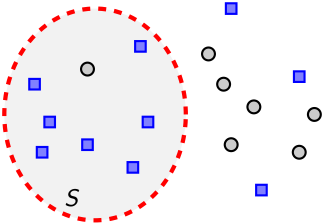

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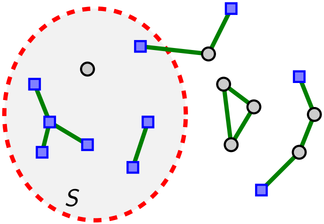

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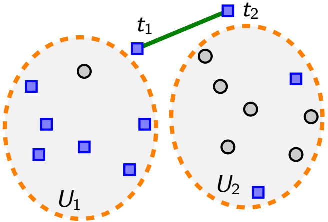

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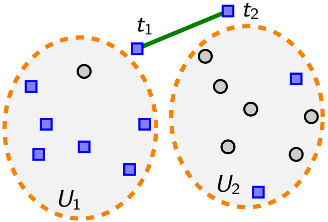

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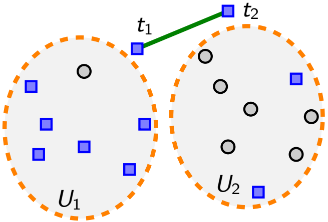

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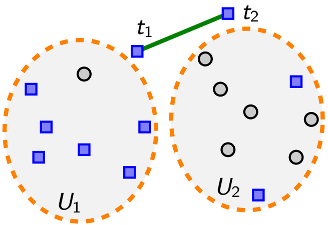

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Other candidates for investigation:

- Stable-set polytopes of claw-free graphs (current work with Gianpaolo Oriolo and Gautier Stauffer).
- Stable-set polytopes of perfect graphs (polyhedral description is known, but best (known) extended formulation has $\mathcal{O}(n^{\log n})$ facets).