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- Ground set $E$ (finite)
- Feasible solutions $\mathcal{F} \subseteq 2^E$
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Theorem (Schulz, Weismantel & Ziegler, 1995; Grötschel & Lovász, 1995)

We can solve the augmentation problem (for arbitrary objective vectors) in polynomial time if and only if we can solve the optimization problem (for arbitrary objective vectors) in polynomial time.
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Polyhedral Approach: Optimization

Polyhedral method:

- Identify \( F \in \mathcal{F} \) with \( \chi(F) \in \{0, 1\}^E \) s.t. \( \chi(F)_e = 1 \iff e \in F \).
Polyhedral Approach: Optimization

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- Optimization problem is then to minimize $\langle c, x \rangle$ over $x \in X$. 

\[
\begin{array}{cccccccc}
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- Optimization problem is then to minimize $\langle c, x \rangle$ over $x \in \text{conv}(X)$.
- Find an outer description of $\text{conv}(X)$, i.e., $\text{conv}(X) = \{x \in \mathbb{R}^E : Ax \leq b\}$.
- Optimization problem is now an LP and we can use black-box solvers.
One drawback of the polyhedral method:

- Consider \( X := \{ x \in \{0, 1\}^n : \sum_{i=1}^n \text{ even} \} \).
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- $P = \text{conv}(X)$ has many facets, but maybe there exists an extension $(Q, \pi)$ ($Q \subseteq \mathbb{R}^d$ polyhedron, $\pi : \mathbb{R}^d \to \mathbb{R}^n$ linear with $P = \pi(Q)$) with few facets?
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\[\text{Theorem (Balas, 1979)}\]

Let $P_1, \ldots, P_k \subseteq \mathbb{R}^n$ be polytopes. Then
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For parity polytope:
- $\text{conv}(X) = \text{conv}(\bigcup_{k \text{ even}} \{x \in [0,1]^n : \sum_{i=1}^n = k\})$
- Applying the theorem: $xc(\text{conv}(X)) \leq O(n^2)$
Hard problems:

- Max-Cut problem: cut polytope for $K_n$ (complete graph with $n$ nodes) has extension complexity $2^{\Omega(n)}$ (Fiorini, Massar, Pokutta, Tiwary & de Wolf, 2012), best bound is $1.5^n$ (Kaibel & Weltge, 2013).
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  - If $F$ is face of $P$, then $\chi_c(F) \leq \chi_c(P)$.
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Matching:

- A perfect matching in a graph $G = (V, E)$ is a set $M \subseteq E$ with $|M \cap \delta(v)| = 1$.

- The weighted perfect matching problem can be solved in polynomial time (Edmonds, 1965).
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Theorem (Rothvoss, 2013)

For every even $n$, $xc(P_{\text{pmatch}}(n)) \geq 2^{\Omega(n)}$. Here, $P_{\text{pmatch}}(n)$ denotes the perfect matching polytope for $K_n$. 
Polyhedral Approach: Augmentation

Polyhedral version of the augmentation problem:

- Consider a polyhedron $P = \{ x \in \mathbb{R}^n : Ax \leq b \}$ and an objective vector $c \in \mathbb{R}^n$.
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- The polyhedron for this task is the radial cone:

\[
K_P(v) := \text{cone}(P - v) + v \\
= \{ x \in \mathbb{R}^n : A_{i,\ast}x \leq b_i \text{ for all } i \text{ with } A_{\ast,i}v = b_i \}
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Nice problems:

- For $v \in P$ we have $xc(K_P(v)) \leq xc(P)$.
- Consequence: nice polyhedra have nice radial cones.
Radial Cones: Basic results

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- Braun, Fiorini, Pokutta & Steurer showed that also the cut cone (radial cone of the cut polytope at vertex $\emptyset$) has exponential extension complexity.
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What remains?
- Matching polytopes & friends (this talk).
- Stable-set polytopes of claw-free or perfect graphs.
Definitions \((K_n = (V_n, E_n))\) complete graph on \(n\) nodes; \(T \subseteq V\), \(|T|\) even):

- \(J \subseteq E\) is a \(T\)-join if \(\left|J \cap \delta(v)\right|\) is odd \(\iff v \in T\)

- \(C = \delta(S) \subseteq E\) is a \(T\)-cut if \(\left|S \cap T\right|\) is odd.
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**Facts:**

- Both minimization problems can be solved in polynomial time for \(c \geq \emptyset\).
- Each \(T\)-join \(J\) intersects each \(T\)-cut \(C\) in at least one edge:
  \[|J \cap C| = \langle \chi(J), \chi(C) \rangle \geq 1\]
\textbf{T-Join- and T-Cut-Polyhedra}

\textit{Polyhedra} (Edmonds & Johnson, 1973):

\begin{itemize}
  \item \textit{T}-join Polyhedron $P_{T\text{-join}}(n)^{\uparrow}$:
    \begin{align*}
    \langle \chi(C), x \rangle & \geq 1 \quad \text{for each } T\text{-cut } C \\
    x_e & \geq 0 \quad \text{for each } e \in E
    \end{align*}

  \item \textit{T}-cut Polyhedron $P_{T\text{-cut}}(n)^{\uparrow}$:
    \begin{align*}
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  \langle \chi(C), x \rangle \geq 1 \quad \text{for each } T\text{-cut } C
  \]
  \[
  x_e \geq 0 \quad \text{for each } e \in E
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- $T$-cut Polyhedron $P_{T\text{-cut}}(n)^\uparrow$:
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**Relation to perfect matchings:**

- A $T$-join $J \subseteq E$ is a perfect matching on nodes $T$ if and only if $x = \chi(J)$ satisfies the valid inequalities $x_e \geq 0$ for all $e \in E \setminus E[T]$ and $\sum_{e \in \delta(v)} x_e \geq 1$ for all $v \in T$ with equality.
T-Join- and T-Cut-Polyhedra

Polyhedra (Edmonds & Johnson, 1973):

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**T-Join- and T-Cut-Polyhedra**

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**T-Join- and T-Cut-Polyhedra**

**Polyhedra** (Edmonds & Johnson, 1973):

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- Consequence:
  \[
  xc(\text{P}_{T-\text{join}}(n)^\uparrow) \geq 2^\Omega(|T|)
  \]

**Proposition (Walter & Weltge, 2018)**

*For every $n$ and every set $T \subseteq V_n$, $xc(\text{P}_{T-\text{join}}(n)^\uparrow) \leq O(n^2 \cdot 2^{|T|})$.***
Blocking Polarity: Basics

Definitions:

- A polyhedron $P \subseteq \mathbb{R}^d_+$ is blocking if $x' \geq x$ implies $x' \in P$ for all $x \in P$.
- Possible descriptions are:

\[
P = \{ x \in \mathbb{R}^d_+ : \langle y^{(i)}, x \rangle \geq 1 \text{ for } i = 1, \ldots, m \} \quad (y^{(1)}, \ldots, y^{(m)} \in \mathbb{R}^d_+) \\
P = \text{conv}\{x^{(1)}, \ldots, x^{(k)}\} + \mathbb{R}^d_+ \quad (x^{(1)}, \ldots, x^{(k)} \in \mathbb{R}^d_+)
\]

\[\text{P} \]

\[x_1 \geq 0\]
\[x_2 \geq 0\]
Blocking Polarity: Basics

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- The **blocker** of $P$ is defined via $B(P) := \{ y \in \mathbb{R}_+^d : \langle x, y \rangle \geq 1 \text{ for all } x \in P \}$. 

![](image_of_diagram)
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![Graph of Blocking Polarity](image)
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Blocking Polarity: Basics

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![Diagram of blocking polyhedron and its blocker](image.png)
Blocking Polarity: Basics

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\[\begin{align*}
  x_1 &\geq 0 \\
  \left(\frac{1}{5}, \frac{2}{5}\right)^\top &\leq 0 \\
  x_2 &\geq 0 \\
  (0, 1) &\leq 1 \\
  (1, 0) &\leq 1 \\
  \left(\frac{1}{5}, \frac{2}{5}\right) &\leq 1
\end{align*}\]
Blocking Polarity: Basics

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![Diagram of Blocking Polarity](image)
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- The blocker of $P$ is defined via $B(P) := \{ y \in \mathbb{R}_+^d : \langle x, y \rangle \geq 1 \text{ for all } x \in P \}$.
- If $P$ is blocking, then $B(B(P)) = P$. 

![Diagram](image_url)
Proposition (Martin, 1991; Conforti, Kaibel, Walter & Weltge, 2015)

Given a non-empty polyhedron \( Q \) and \( \gamma \in \mathbb{R} \), let

\[
P := \{ x : \langle y, x \rangle \geq \gamma \text{ for all } y \in Q \}.
\]

Then \( xc(P) \leq xc(Q) + 1 \).

Proof idea:

- Separation problem for inequalities is a linear program.
- Apply strong LP duality.
Proposition (Martin, 1991; Conforti, Kaibel, Walter & Weltge, 2015)

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Consequences:

- $xc(B(P))$ and $xc(P)$ differ by at most $d$. 
Blocking Polarity: Extensions

Proposition (Martin, 1991; Conforti, Kaibel, Walter & Weltge, 2015)

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Proof idea:
- Separation problem for inequalities is a linear program.
- Apply strong LP duality.

Consequences:
- $xc(B(P))$ and $xc(P)$ differ by at most $d$.
- $2^{\Omega(|T|)} \leq xc(P_{T\text{-cut}}(n)) \leq O(n^2 \cdot 2^{|T|})$. 
Polar object of radial cone:

- Any $v \in P$ defines a face $F_{B(P)}(v) := \{ y \in B(P) : \langle v, y \rangle = 1 \}$ of $B(P)$.
Blocking Polarity: Radial Cones

Polar object of radial cone:
- Any \( v \in P \) defines a face \( F_{B(P)}(v) := \{ y \in B(P) : \langle v, y \rangle = 1 \} \) of \( B(P) \).

Lemma

Let \( P \subseteq \mathbb{R}^d_+ \) be a blocking polyhedron and let \( v \in P \).

(i) \( F_{B(P)}(v) = \{ y \in \mathbb{R}^d : \langle v, y \rangle = 1, \langle x, y \rangle \geq 1 \ \forall x \in K_P(v) \} \).

(ii) \( K_P(v) = \{ x \in \mathbb{R}^d : \langle y, x \rangle \geq 1 \ \forall y \in F_{B(P)}(v) \} \).
Blocking Polarity: Radial Cones

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Lemma

Let \( P \subseteq \mathbb{R}^d_+ \) be a blocking polyhedron and let \( v \in P \).

(i) \( F_{B(P)}(v) = \{ y \in \mathbb{R}^d : \langle v, y \rangle = 1, \langle x, y \rangle \geq 1 \forall x \in K_P(v) \} \).

(ii) \( K_P(v) = \{ x \in \mathbb{R}^d : \langle y, x \rangle \geq 1 \forall y \in F_{B(P)}(v) \} \).

Consequence:
- \( xc(K_P(v)) \) and \( xc(F_{B(P)}(v)) \) differ by at most 1.
- To prove lower or upper bounds on \( xc(K_P(v)) \), analyze \( F_{B(P)}(v) \)!
**Theorem (Ventura & Eisenbrand, 2003)**

For every set $T \subseteq V_n$ with $|T|$ even and every vertex $v$ of $P_{T\text{-join}}(n)^\uparrow$, corresponding to a $T$-join $J \subseteq E_n$ in $K_n$, the extension complexity of the radial cone of $P_{T\text{-join}}(n)$ at $v$ is most $\mathcal{O}(\frac{|J| \cdot n^2}{|T|})$.

**Their proof:** ad-hoc construction using sets of flow variables.
Radial Cones of $T$-Join Polyhedra

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For every set $T \subseteq V_n$ with $|T|$ even and every vertex $v$ of $P_{T\text{-join}}(n)^\uparrow$, corresponding to a $T$-join $J \subseteq E_n$ in $K_n$, the extension complexity of the radial cone of $P_{T\text{-join}}(n)$ at $v$ is most $O(|J| \cdot n^2)$.

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Our new proof:
Theorem (Ventura & Eisenbrand, 2003)

For every set $T \subseteq V_n$ with $|T|$ even and every vertex $v$ of $P_{T\text{-join}}(n)^\uparrow$, corresponding to a $T$-join $J \subseteq E_n$ in $K_n$, the extension complexity of the radial cone of $P_{T\text{-join}}(n)$ at $v$ is most $O(|J| \cdot n^2)$.

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**Our new proof:**

- By Lemma, theorem reduces to $xc(P)$ for

$$P := \left\{ x \in P_{T\text{-cut}}(n)^\uparrow : \sum_{e \in J} x_e = 1 \right\}.$$
Radial Cones of $T$-Join Polyhedra

Theorem (Ventura & Eisenbrand, 2003)

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- For each $m \in J$, let $F_m$ be the face of $P$ with $x_m = 1$ (and $x_e = 0 \ \forall \ e \in J \setminus \{ m \}$).
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Our new proof:

- By Lemma, theorem reduces to $\text{xc}(P)$ for

$$P := \left\{ x \in P_{T\text{-cut}}(n)^\uparrow : \sum_{e \in J} x_e = 1 \right\}.$$

- For each $m \in J$, let $F_m$ be the face of $P$ with $x_m = 1$ (and $x_e = 0 \; \forall e \in J \setminus \{m\}$).

- But $F_m$ is also a face of $P_{T'/\text{-cut}}(n)^\uparrow$ for $T' = m$ (set containing the nodes).
Radial Cones of $T$-Join Polyhedra

Theorem (Ventura & Eisenbrand, 2003)

For every set $T \subseteq V_n$ with $|T|$ even and every vertex $v$ of $P_{T\text{-join}}(n)^\uparrow$, corresponding to a $T$-join $J \subseteq E_n$ in $K_n$, the extension complexity of the radial cone of $P_{T\text{-join}}(n)$ at $v$ is most $O(|J| \cdot n^2)$.

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- We obtain $xc(F_m) \leq O(n^2 \cdot 2^{|T'|}) = O(n^2)$. 
Theorem (Ventura & Eisenbrand, 2003)

For every set $T \subseteq V_n$ with $|T|$ even and every vertex $v$ of $P_{T-join}(n)^\uparrow$, corresponding to a $T$-join $J \subseteq E_n$ in $K_n$, the extension complexity of the radial cone of $P_{T-join}(n)$ at $v$ is most $O(|J| \cdot n^2)$.

Their proof: ad-hoc construction using sets of flow variables.

Our new proof:

- By Lemma, theorem reduces to $xc(P)$ for

$$P := \left\{ \chi \in P_{T-cut}(n)^\uparrow : \sum_{e \in J} \chi_e = 1 \right\}.$$ 

- For each $m \in J$, let $F_m$ be the face of $P$ with $x_m = 1$ (and $x_e = 0 \ \forall \ e \in J \setminus \{m\}$).

- But $F_m$ is also a face of $P_{T'-cut}(n)^\uparrow$ for $T' = m$ (set containing the nodes).

- We obtain $xc(F_m) \leq O(n^2 \cdot 2^{|T'|}) = O(n^2)$.

- $P$ is convex hull of union of all $F_m$. 

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Matthias Walter

Extended Formulations for Radial Cones

Kolkom 2018

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Theorem (Walter & Weltge, 2018)

For sets \( T \subseteq V_n \) with \( |T| \) even and vertices \( v \) of \( P_{T\text{-cut}}(n)^\uparrow \), the extension complexity of the radial cone of \( P_{T\text{-cut}}(n) \) at \( v \) is least \( 2^{\Omega(|T|)} \).
Theorem (Walter & Weltge, 2018)

For sets $T \subseteq V_n$ with $|T|$ even and vertices $v$ of $P_{T\text{-cut}}(n)^\uparrow$, the extension complexity of the radial cone of $P_{T\text{-cut}}(n)$ at $v$ is least $2^{\Omega(|T|)}$.

Proof:
Radial Cones of $T$-Cut Polyhedra

Theorem (Walter & Weltge, 2018)

For sets $T \subseteq V_n$ with $|T|$ even and vertices $v$ of $P_{T\text{-cut}}(n)^\uparrow$, the extension complexity of the radial cone of $P_{T\text{-cut}}(n)$ at $v$ is least $2^{\Omega(|T|)}$.

Proof:

- Let $v = \chi(\delta(S))$. 

Radial Cones of $T$-Cut Polyhedra

**Theorem (Walter & Weltge, 2018)**

For sets $T \subseteq V_n$ with $|T|$ even and vertices $v$ of $P_{T\text{-cut}}(n)^\uparrow$, the extension complexity of the radial cone of $P_{T\text{-cut}}(n)$ at $v$ is least $2^{\Omega(|T|)}$.

**Proof:**

- Let $v = \chi(\delta(S))$.
- By Lemma, theorem reduces to $xc(P)$ for

$$P := \left\{ x \in P_{T\text{-join}}(n)^\uparrow : \sum_{e \in \delta(S)} x_e = 1 \right\}.$$
Radial Cones of $T$-Cut Polyhedra

Theorem (Walter & Weltge, 2018)

For sets $T \subseteq V_n$ with $|T|$ even and vertices $v$ of $P_{T\text{-cut}}(n)^\dagger$, the extension complexity of the radial cone of $P_{T\text{-cut}}(n)$ at $v$ is least $2^{\Omega(|T|)}$.

Proof:

- Let $v = \chi(\delta(S))$.
- By Lemma, theorem reduces to $xc(P)$ for

$$P := \left\{ x \in P_{T\text{-join}}(n)^\dagger : \sum_{e \in \delta(S)} x_e = 1 \right\}.$$

- Let $t_1 \in S$, $t_2 \in V_n \setminus S$ as well as

$$U_1 := S \setminus \{t_1\}, \ U_2 := (V_n \setminus (S \cup \{t_2\})).$$
Radial Cones of $T$-Cut Polyhedra

**Theorem (Walter & Weltge, 2018)**

For sets $T \subseteq V_n$ with $|T|$ even and vertices $v$ of $P_{\text{cut}}(n)^\uparrow$, the extension complexity of the radial cone of $P_{\text{cut}}(n)$ at $v$ is least $2^{\Omega(|T|)}$.

**Proof:**

- Let $v = \chi(\delta(S))$.
- By Lemma, theorem reduces to $xc(P)$ for
  
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  $$U_1 := S \setminus \{ t_1 \}, \quad U_2 := (V_n \setminus (S \cup \{ t_2 \})).$$

- Let $F$ be the face of $P$ with $x_{\{t_1,t_2\}} = 1$ and $x_e = 0$ for all edges between $U_1$, $U_2$ and $\{ t_1, t_2 \}$. 

\[\text{Diagram of $U_1$ and $U_2$ with vertices $t_1$ and $t_2$.}\]
Theorem (Walter & Weltge, 2018)

For sets \( T \subseteq V_n \) with \( |T| \) even and vertices \( v \) of \( P_T\text{-cut}(n) \), the extension complexity of the radial cone of \( P_T\text{-cut}(n) \) at \( v \) is least \( 2^{\Omega(|T|)} \).

Proof:

- Let \( v = \chi(\delta(S)) \).
- By Lemma, theorem reduces to \( xc(P) \) for
  \[
  P := \left\{ x \in P_{T\text{-join}}(n) : \sum_{e \in \delta(S)} x_e = 1 \right\}.
  \]
- Let \( t_1 \in S, t_2 \in V_n \setminus S \) as well as
  \( U_1 := S \setminus \{t_1\}, U_2 := (V_n \setminus (S \cup \{t_2\})) \).
- Let \( F \) be the face of \( P \) with \( x_{\{t_1,t_2\}} = 1 \) and \( x_e = 0 \) for all edges between \( U_1, U_2 \) and \( \{t_1,t_2\} \).
- \( F \) ist a Cartesian product of a vector and two \((T \cap U_i)\)-join polyhedra on \( U_i \) for \( i = 1, 2 \), where \( |T_1| + |T_2| = |T| - 2 \).
Radial Cones of $T$-Cut Polyhedra

**Theorem (Walter & Weltge, 2018)**

For sets $T \subseteq V_n$ with $|T|$ even and vertices $v$ of $P_{T\text{-cut}}(n)^\uparrow$, the extension complexity of the radial cone of $P_{T\text{-cut}}(n)$ at $v$ is least $2^{\Omega(|T|)}$.

**Proof:**

- Let $v = \chi(\delta(S))$.
- By Lemma, theorem reduces to $xc(P)$ for

$$P := \left\{ x \in P_{\text{T-join}}(n)^\uparrow : \sum_{e \in \delta(S)} x_e = 1 \right\}.$$

- Let $t_1 \in S$, $t_2 \in V_n \setminus S$ as well as $U_1 := S \setminus \{t_1\}$, $U_2 := (V_n \setminus (S \cup \{t_2\}))$.
- Let $F$ be the face of $P$ with $x_{\{t_1,t_2\}} = 1$ and $x_e = 0$ for all edges between $U_1$, $U_2$ and $\{t_1,t_2\}$.
- $F$ is a Cartesian product of a vector and two $(T \cap U_i)$-join polyhedra on $U_i$ for $i = 1, 2$, where $|T_1| + |T_2| = |T| - 2$.
- We obtain $xc(P) \geq xc(F) \geq 2^{\Omega(|T_i|)}$ for $i = 1, 2$. 

![Diagram](image-url)
Thanks!

Conclusion:

- Extended formulations can help, but only sometimes.
- Although polynomially solvable, there is no obvious way to solve the minimum-weight $T$-cut problem with LP techniques.
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Other candidates for investigation:

- Stable-set polytopes of claw-free graphs (current work with Gianpaolo Oriolo and Gautier Stauffer).
- Stable-set polytopes of perfect graphs (polyhedral description is known, but best (known) extended formulation has $O(n \log n)$ facets).