Extended Formulations for Radial Cones

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Joint work with

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Operations Research | RWTH AACHEN UNIVERSITY

Combinatorial optimization problem:

- Ground set E (finite)
- Feasible solutions $\mathcal{F} \subseteq 2^E$
- Objective vector $c \in \mathbb{Q}^E$
- Goal: minimize cost $c(F) := \sum_{e \in F} c_e$ over all $F \in \mathcal{F}$.

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- Given $F \in \mathcal{F}$, determine optimality or find $F' \in \mathcal{F}$ with $c(F') < c(F)$.

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Theorem (Schulz, Weismantel & Ziegler, 1995; Grötschel & Lovász, 1995)

We can solve the augmentation problem (for arbitrary objective vectors) in polynomial time if and only if we can solve the optimization problem (for arbitrary objective vectors) in polynomial time.

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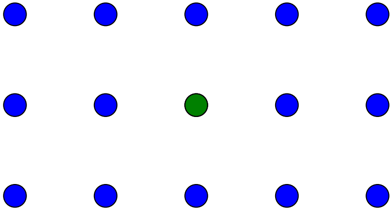
Idea:

- Suppose $c \in \{0,1\}^E$, how many augmentation steps will you need?
- Apply bit scaling.

Polyhedral Approach: Optimization

Polyhedral method:

- Identify $F \in \mathcal{F}$ with $\chi(F) \in \{0,1\}^E$ s.t. $\chi(F)_e = 1 \iff e \in F$.



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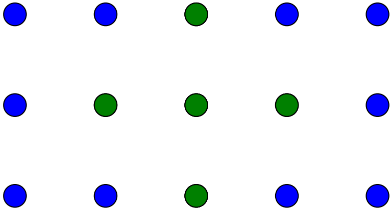
- Optimization problem is then to minimize $\langle c, x \rangle$ over $x \in X$.



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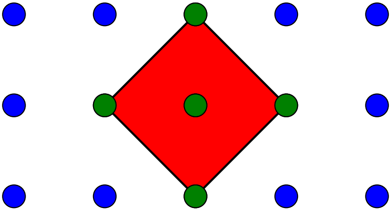

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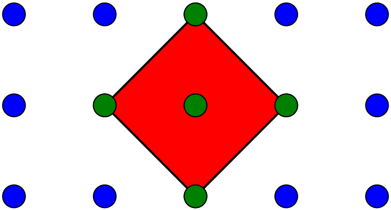
- Optimization problem is then to minimize $\langle c, x \rangle$ over $x \in \text{conv}(X)$.
- Find an outer description of $\text{conv}(X)$, i.e., $\text{conv}(X) = \{x \in \mathbb{R}^E : Ax \leq b\}$.
- Optimization problem is now an LP and we can use black-box solvers.[1]



[1] . . . or devise primal-dual algorithms.

One drawback of the polyhedral method:

- Consider $X := \{x \in \{0,1\}^n : \sum_{i=1}^n \text{ even}\}$.
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- Inequality description (Jeroslow, 1975) requires 2^{n-1} inequalities:

$$\sum_{i \in I} (1 - x_i) + \sum_{i \notin I} x_i \geq 1 \text{ for all } I \subseteq [n] \text{ with } |I| \text{ odd}$$

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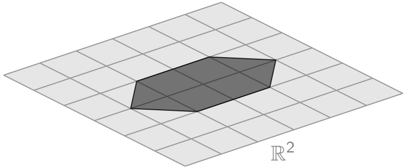
- $P = \text{conv}(X)$ has many facets, but maybe there exists an extension (Q, π) ($Q \subseteq \mathbb{R}^d$, $\pi : \mathbb{R}^d \to \mathbb{R}^n$ linear with $P = \pi(Q)$) with few facets?



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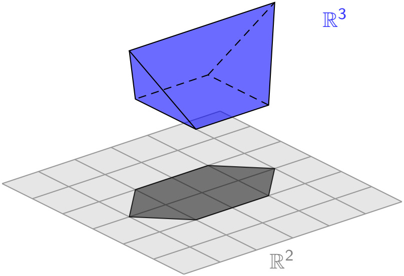

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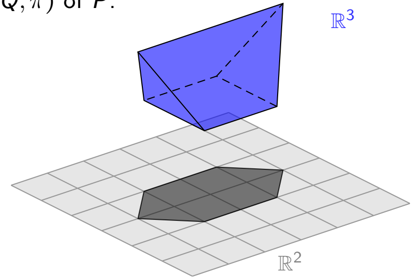

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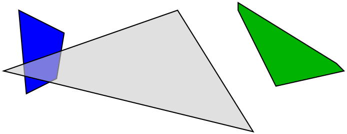
Let $P_1, \ldots, P_k \subseteq \mathbb{R}^n$ be polytopes. Then $\text{xc}(P_1 \cup \cdots \cup P_k) \leq \sum_{i=1}^k (\text{xc}(P_i) + 1)$.

Disjunctive programming:



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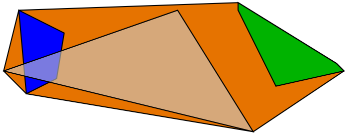

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- Applying the theorem: $\text{xc}(\text{conv}(X)) \leq \mathcal{O}(n^2)$

Limitations

Hard problems:

- Max-Cut problem: cut polytope for K_n (complete graph with n nodes) has extension complexity $2^{\Omega(n)}$ (Fiorini, Massar, Pokutta, Tiwary & de Wolf, 2012), best bound is 1.5^n (Kaibel & Weltge, 2013).

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- A perfect matching in a graph $G = (V, E)$ is a set $M \subseteq E$ with $|M \cap \delta(v)| = 1$.
- The weighted perfect matching problem can be solved in polynomial time (Edmonds, 1965).

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Theorem (Rothvoss, 2013)

For every even n, $\text{xc}(P_{pmatch}(n)) \geq 2^{\Omega(n)}$. Here, $P_{pmatch}(n)$ denotes the perfect matching polytope for K_n.

Polyhedral version of the augmentation problem:

- Consider a polyhedron $P = \{x \in \mathbb{R}^n : Ax \le b\}$ and an objective vector $c \in \mathbb{R}^n$.
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- The polyhedron for this task is the radial cone:

$$K_P(v) := \text{cone}(P - v) + v$$
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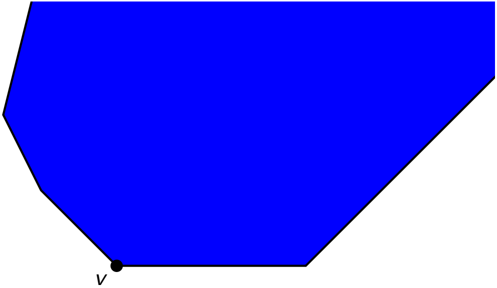

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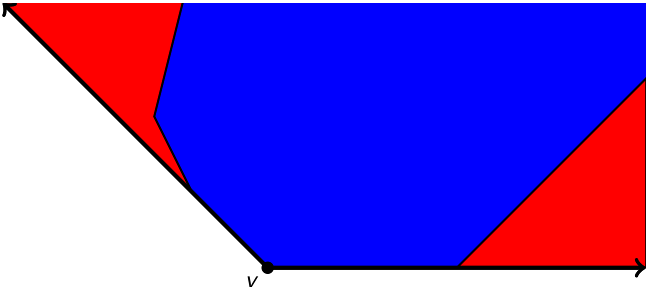

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- Extension complexity of radial cones is inherited to projections and faces.
- Consequence: exponential lower bounds for your favorite polytopes (TSP, Stable set, 3d matching, etc.) that correspond to hard problems.

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- Consequence: nice polyhedra have nice radial cones.

Hard problems:

- Braun, Fiorini, Pokutta & Steurer showed that also the cut cone (radial cone of the cut polytope at vertex $\mathbb{O}$) has exponential extension complexity.
- Extension complexity of radial cones is inherited to projections and faces.
- Consequence: exponential lower bounds for your favorite polytopes (TSP, Stable set, 3d matching, etc.) that correspond to hard problems.

What remains?

- Matching polytopes & friends (this talk).
- Stable-set polytopes of claw-free or perfect graphs.

T-Joins & T-Cuts

Definitions ($K_n = (V_n, E_n)$ complete graph on n nodes; $T \subseteq V$, $|T|$ even)**:**

- $J \subseteq E$ is a T-join if $|J \cap \delta(v)|$ is odd $\iff v \in T$
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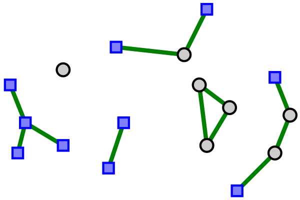

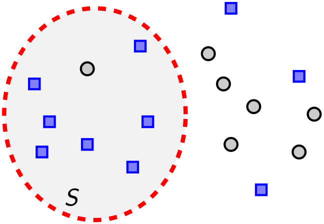

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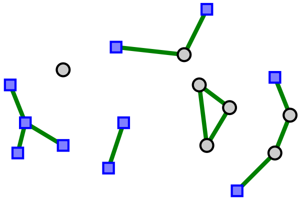

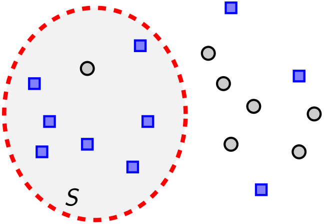

Facts:

- Both minimization problems can be solved in polynomial time for $c \geq \mathbb{O}$.
- Each T-join J intersects each T-cut C in at least one edge:

$$|J \cap C| = \langle \chi(J), \chi(C) \rangle \geq 1$$

T-Join- and T-Cut-Polyhedra

Polyhedra (Edmonds & Johnson, 1973)**:**

- T-join Polyhedron $P_{T\text{-join}}(n)^{\uparrow}$:

$$\begin{aligned} \langle \chi(C), x \rangle &\geq 1 \quad \text{for each } T\text{-cut } C \\ x_e &\geq 0 \quad \text{for each } e \in E \end{aligned}$$

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Relation to perfect matchings:

- A T-join $J \subseteq E$ is a perfect matching on nodes T if and only if $x = \chi(J)$ satisfies the valid inequalities $x_e \geq 0$ for all $e \in E \setminus E[T]$ and $\sum_{e \in \delta(v)} x_e \geq 1$ for all $v \in T$ with equality.

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Proposition (Walter & Weltge, 2018)

For every n and every set $T \subseteq V_n$, $\text{xc}(P_{T\text{-join}}(n)^{\uparrow}) \leq \mathcal{O}(n^2 \cdot 2^{|T|})$.

Idea:

- For each $S \subseteq T$ with $|S| = \frac{1}{2}|T|$, consider the b-flow polyhedron for $b_v = -1$ for all $v \in S$, $b_v = 1$ for all $v \in T \setminus S$ and $b_v = 0$ otherwise.
- Apply disjunctive programming over all such polyhedra.

Blocking Polarity: Basics

Definitions:

- A polyhedron $P \subseteq \mathbb{R}^d_+$ is blocking if $x' \geq x$ implies $x' \in P$ for all $x \in P$.
- Possible descriptions are:

$$P = \{x \in \mathbb{R}^d_+ : \langle y^{(i)}, x \rangle \geq 1 \text{ for } i = 1, \dots, m\} \qquad (y^{(1)}, \dots, y^{(m)} \in \mathbb{R}^d_+)$$

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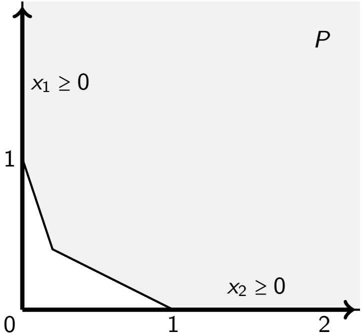

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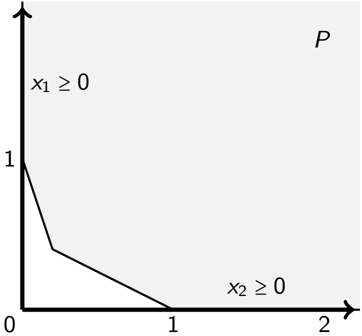

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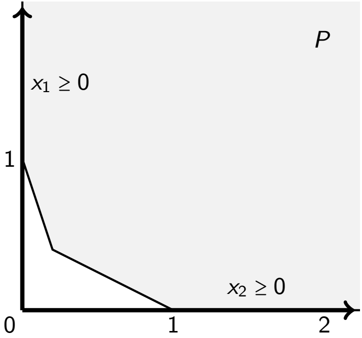

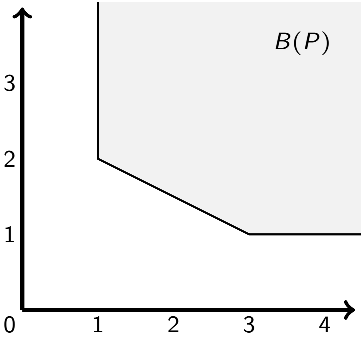

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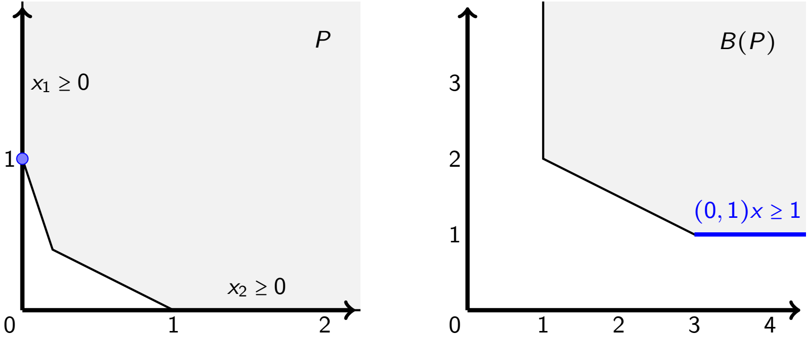

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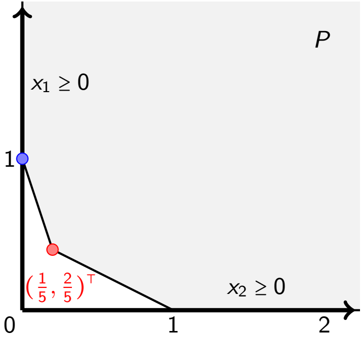

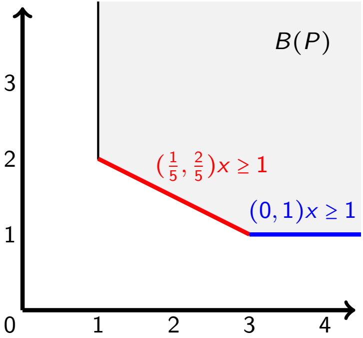

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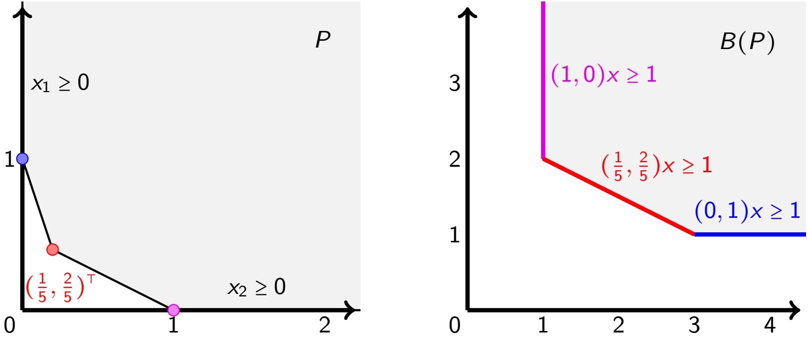

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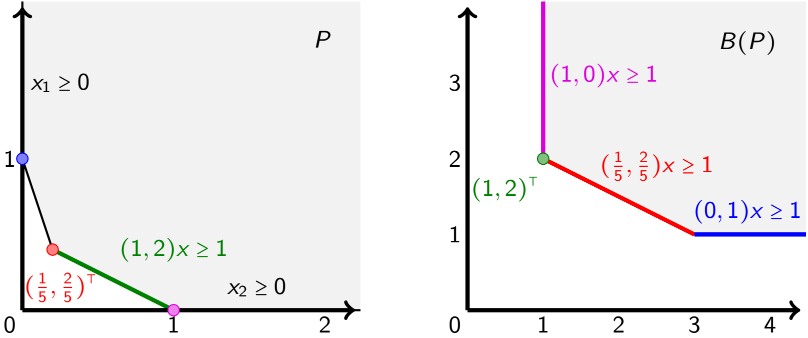

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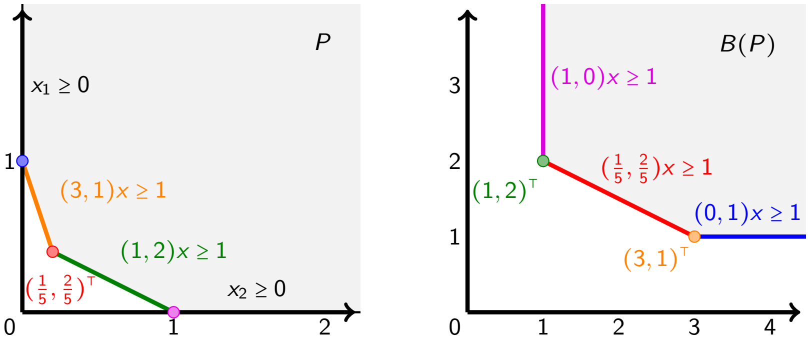

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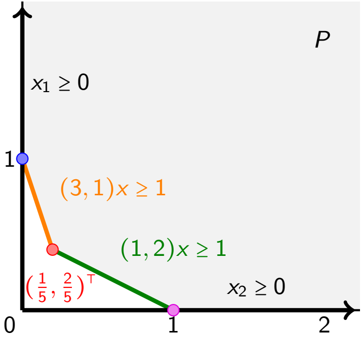

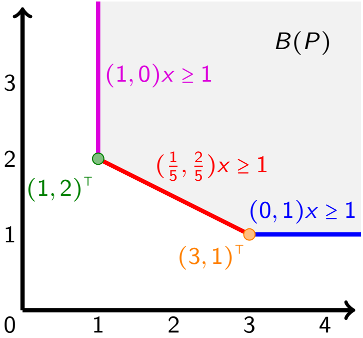

Proposition (Martin, 1991; Conforti, Kaibel, Walter & Weltge, 2015)

Given a non-empty polyhedron Q and $\gamma \in \mathbb{R}$, let

$$P := \{x : \langle y, x \rangle \geq \gamma \text{ for all } y \in Q\}.$$

Then $\text{xc}(P) \leq \text{xc}(Q) + 1$.

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Consequences:

- $\text{xc}(B(P))$ and $\text{xc}(P)$ differ by at most d.
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Lemma

Let $P \subseteq \mathbb{R}^d_+$ be a blocking polyhedron and let $v \in P$.

(i) $F_{B(P)}(v) = \left\{y \in \mathbb{R}^d : \langle v, y \rangle = 1,\ \langle x, y \rangle \geq 1\ \forall x \in K_P(v)\right\}$.

(ii) $K_P(v) = \left\{x \in \mathbb{R}^d : \langle y, x \rangle \geq 1\ \forall y \in F_{B(P)}(v)\right\}$.

Proof:

- Elementary convex geometry

Polar object of radial cone:

- Any $v \in P$ defines a face $F_{B(P)}(v) := \{y \in B(P) : \langle v, y \rangle = 1\}$ of $B(P)$.

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Consequence:

- $\text{xc}(K_P(v))$ and $\text{xc}(F_{B(P)}(v))$ differ by at most 1.
- To prove lower or upper bounds on $\text{xc}(K_P(v))$, it suffices to do analyze the face $F_{B(P)}(v)$ of $B(P)$.

Radial Cones of T-Join Polyhedra

Theorem (Ventura & Eisenbrand, 2003)

For every set $T \subseteq V_n$ with $|T|$ even and every vertex v of $P_{T\text{-join}}(n)^{\uparrow}$, corresponding to a T-join $J \subseteq E_n$ in K_n, the extension complexity of the radial cone of $P_{T\text{-join}}(n)$ at v is most $\mathcal{O}(|J| \cdot n^2)$.

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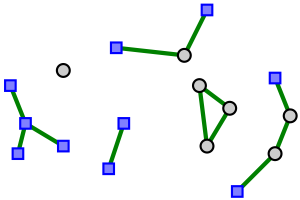

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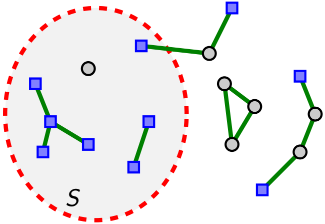

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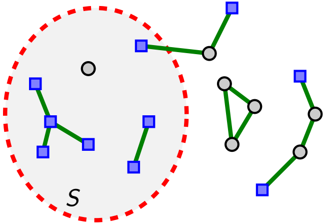

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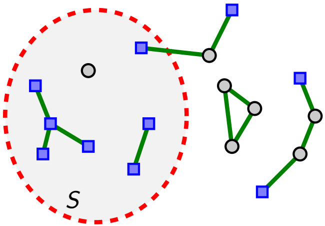

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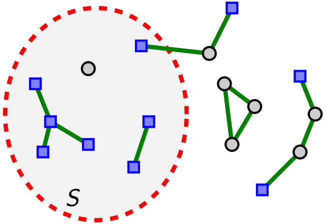

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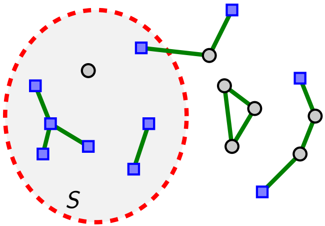
- P is convex hull of union of all F_m.



Radial Cones of T-Cut Polyhedra

Theorem (Walter & Weltge, 2018)

For sets $T \subseteq V_n$ with $|T|$ even and vertices v of $P_{T\text{-cut}}(n)^{\uparrow}$, the extension complexity of the radial cone of $P_{T\text{-cut}}(n)$ at v is least $2^{\Omega(|T|)}$.

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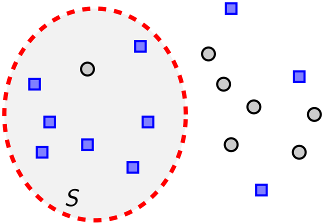

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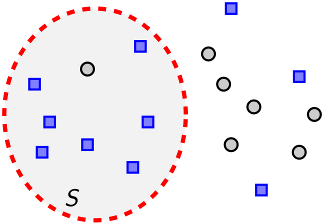

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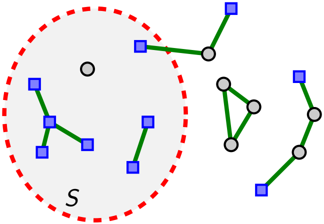

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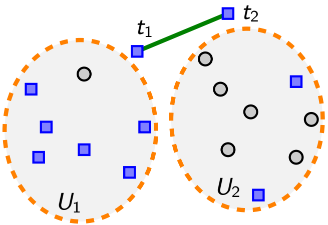

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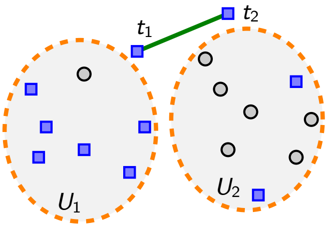

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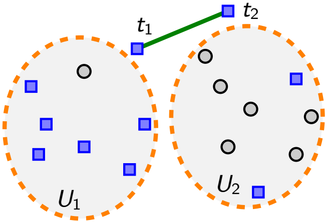

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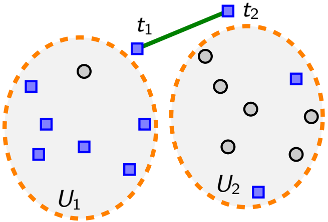

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Other candidates for investigation:

- Stable-set polytopes of claw-free graphs (current work with Gianpaolo Oriolo and Gautier Stauffer).
- Stable-set polytopes of perfect graphs (polyhedral description is known, but best extended formulation has $\mathcal{O}(n^{\log n})$ facets).