Extended Formulations for Radial Cones

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Joint work with

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IMO Oberseminar, Magdeburg, 02.11.2018
Combinatorial optimization problem:

- Ground set $E$ (finite)
- Feasible solutions $\mathcal{F} \subseteq 2^E$
- Objective vector $c \in \mathbb{Q}^E$
- Goal: minimize cost $c(F) := \sum_{e \in F} c_e$ over all $F \in \mathcal{F}$. 
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Augmentation problem:

- Given $F \in \mathcal{F}$, determine optimality or find $F' \in \mathcal{F}$ with $c(F') < c(F)$. 

Theorem (Schulz, Weismantel & Ziegler, 1995; Grötschel & Lovász, 1995)

We can solve the augmentation problem (for arbitrary objective vectors) in polynomial time if and only if we can solve the optimization problem (for arbitrary objective vectors) in polynomial time.

Idea:

Suppose $c \in \{0, 1\}^E$, how many augmentation steps will you need?

Apply bit scaling.
Optimization & Augmentation

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- Apply bit scaling.
Polyhedral Approach: Optimization

Polyhedral method:
- Identify $F \in \mathcal{F}$ with $\chi(F) \in \{0, 1\}^E$ s.t. $\chi(F)_e = 1 \iff e \in F$. 

\[
X = \{\chi(F) : F \in \mathcal{F}\} \subseteq \{0, 1\}^E.
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Optimization problem is then to minimize $c^T x$ over $x \in X$.

Find an outer description of $\text{conv}(X)$, i.e., $\text{conv}(X) = \{x \in \mathbb{R}^E : Ax \leq b\}$.

Optimization problem is now an LP and we can use black-box solvers.

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- Consider $X := \{ x \in \{0, 1\}^n : \sum_{i=1}^n \text{even} \}$.
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$$\sum_{i \in I} (1 - x_i) + \sum_{i \notin I} x_i \geq 1 \text{ for all } I \subseteq [n] \text{ with } |I| \text{ odd}$$
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- $P = \text{conv}(X)$ has many facets, but maybe there exists an extension $(Q, \pi)$ ($Q \subseteq \mathbb{R}^d$, $\pi : \mathbb{R}^d \to \mathbb{R}^n$ linear with $P = \pi(Q)$) with few facets?
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Let \( P_1, \ldots, P_k \subseteq \mathbb{R}^n \) be polytopes. Then \( xc(P_1 \cup \cdots \cup P_k) \leq \sum_{i=1}^k (xc(P_i) + 1) \).

Disjunctive programming:
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\[X = \bigcup_{k \text{ even}} \{x \in \{0,1\}^n : \sum_{i=1}^{n} x_i = k\}\]
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- $\text{conv}(X) = \text{conv}( \bigcup_{k \text{ even}} \{ x \in [0, 1]^n : \sum_{i=1}^n = k \} )$.
- Applying the theorem: $xc(\text{conv}(X)) \leq \mathcal{O}(n^2)$.
Limitations

Hard problems:

- Max-Cut problem: cut polytope for $K_n$ (complete graph with $n$ nodes) has extension complexity $2^{\Omega(n)}$ (Fiorini, Massar, Pokutta, Tiwary & de Wolf, 2012), best bound is $1.5^n$ (Kaibel & Weltge, 2013).
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  - If $F$ is face of $P$, then $xc(F) \leq xc(P)$.
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- Based on Karp reductions, write cut polytope as projection of a face of your favorite polytope (TSP, Stable set, 3d matching, etc.).
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**Matching:**
- A perfect matching in a graph $G = (V, E)$ is a set $M \subseteq E$ with $|M \cap \delta(v)| = 1$.
- The weighted perfect matching problem can be solved in polynomial time (Edmonds, 1965).
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Theorem (Rothvoss, 2013)

For every even $n$, $xc(P_{p_{\text{match}}}(n)) \geq 2^{\Omega(n)}$. Here, $P_{p_{\text{match}}}(n)$ denotes the perfect matching polytope for $K_n$. 
Polyhedral version of the augmentation problem:

- Consider a polyhedron $P = \{ x \in \mathbb{R}^n : Ax \leq b \}$ and an objective vector $c \in \mathbb{R}^n$.
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- The polyhedron for this task is the radial cone:

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K_P(v) := \text{cone}(P - v) + v \\
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Radial Cones: Basic results

Nice problems:

- For $v \in P$ we have $xc(K_P(v)) \leq xc(P)$.
- Consequence: nice polyhedra have nice radial cones.
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- Braun, Fiorini, Pokutta & Steurer showed that also the cut cone (radial cone of the cut polytope at vertex \( \emptyset \)) has exponential extension complexity.
- Extension complexity of radial cones is inherited to projections and faces.
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What remains?
- Matching polytopes & friends (this talk).
- Stable-set polytopes of claw-free or perfect graphs.
T-Joins & T-Cuts

Definitions \((K_n = (V_n, E_n))\) complete graph on \(n\) nodes; \(T \subseteq V\), \(|T|\) even:

- \(J \subseteq E\) is a \(T\)-join if 
  \[|J \cap \delta(v)|\text{ is odd} \iff v \in T\]

- \(C = \delta(S) \subseteq E\) is a \(T\)-cut if 
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**Facts:**

- Both minimization problems can be solved in polynomial time for \(c \geq \emptyset\).
- Each \(T\)-join \(J\) intersects each \(T\)-cut \(C\) in at least one edge:
  \[
  |J \cap C| = \langle \chi(J), \chi(C) \rangle \geq 1
  \]
**$T$-Join- and $T$-Cut-Polyhedra**

**Polyhedra** (Edmonds & Johnson, 1973):

- **$T$-join Polyhedron** $P_{T\text{-join}}(n)^\uparrow$:
  
  \[
  \langle \chi(C), x \rangle \geq 1 \quad \text{for each } T\text{-cut } C \\
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\section*{\textbf{T-Join- and T-Cut-Polyhedra}}

\textbf{Polyhedra} (Edmonds & Johnson, 1973):

- \textit{T-join Polyhedron} $P_{T\text{-join}}(n)^\uparrow$:
  \begin{align*}
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\textbf{Relation to perfect matchings:}

- A \textit{T-join} $J \subseteq E$ is a perfect matching on nodes $T$ if and only if $x = \chi(J)$ satisfies the valid inequalities $x_e \geq 0$ for all $e \in E \setminus E[T]$ and $\sum_{e \in \delta(v)} x_e \geq 1$ for all $v \in T$ with equality.
**$T$-Join- and $T$-Cut-Polyhedra**

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**Relation to perfect matchings:**

- A $T$-join $J \subseteq E$ is a perfect matching on nodes $T$ if and only if $x = \chi(J)$ satisfies the valid inequalities $x_e \geq 0$ for all $e \in E \setminus E[T]$ and $\sum_{e \in \delta(v)} x_e \geq 1$ for all $v \in T$ with equality.

- Thus, $P_{T\text{-join}}(n)$ contains $P_{\text{pmatch}}(|T|)$ as a face.
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**Proposition (Walter & Weltge, 2018)**

*For every \(n\) and every set \(T \subseteq V_n\), \(xc(P_{T\text{-join}}(n)^\uparrow) \leq O(n^2 \cdot 2^{|T|})\).*

**Idea:**

- For each \(S \subseteq T\) with \(|S| = \frac{1}{2}|T|\), consider the \(b\)-flow polyhedron for \(b_v = -1\) for all \(v \in S\), \(b_v = 1\) for all \(v \in T \setminus S\) and \(b_v = 0\) otherwise.
- Apply disjunctive programming over all such polyhedra.
Blocking Polarity: Basics

Definitions:

- A polyhedron \( P \subseteq \mathbb{R}_+^d \) is **blocking** if \( x' \geq x \) implies \( x' \in P \) for all \( x \in P \).
- Possible descriptions are:
  
  \[
  P = \{ x \in \mathbb{R}_+^d : \langle y^{(i)}, x \rangle \geq 1 \text{ for } i = 1, \ldots, m \} \quad (y^{(1)}, \ldots, y^{(m)} \in \mathbb{R}_+^d)
  \]
  
  \[
  P = \operatorname{conv}\{ x^{(1)}, \ldots, x^{(k)} \} + \mathbb{R}_+^d \quad (x^{(1)}, \ldots, x^{(k)} \in \mathbb{R}_+^d)
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- The blocker of $P$ is defined via $B(P) := \{ y \in \mathbb{R}_+^d : \langle x, y \rangle \geq 1 \text{ for all } x \in P\}$. 
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- The **blocker** of $P$ is defined via $B(P) := \{ y \in \mathbb{R}^d_+ : \langle x, y \rangle \geq 1 \text{ for all } x \in P \}$.

\[
\begin{align*}
\text{P:} & \quad x_1 \geq 0 \\
& \quad x_2 \geq 0 \\
& \quad \left(\frac{1}{5}, \frac{2}{5}\right) \rangle x \geq 1 \\
\text{B(P):} & \quad (\frac{1}{5}, \frac{2}{5}) \rangle x \geq 1 \\
& \quad (0, 1) \rangle x \geq 1
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Blocking Polarity: Basics

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![Diagram of blocking polyhedron and its blocker](image)
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---

**Diagram:**

- **$P$:**
  
  - $x_1 \geq 0$
  
  - $(3, 1)x \geq 1$
  
  - $(1, 2)x \geq 1$
  
  - $(\frac{1}{5}, \frac{2}{5})^T x \geq 0$
  
  - $x_2 \geq 0$

- **$B(P)$:**
  
  - $(1, 0)x \geq 1$
  
  - $(\frac{1}{5}, \frac{2}{5})x \geq 1$
  
  - $(1, 2)^T$
  
  - $(0, 1)x \geq 1$
  
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- The **blocker** of $P$ is defined via $B(P) := \{ y \in \mathbb{R}^d_+ : \langle x, y \rangle \geq 1 \text{ for all } x \in P \}$.
- If $P$ is blocking, then $B(B(P)) = P$. 

![Diagram of Blocking Polarity](image_url)
Blocking Polarity: Extensions

**Proposition (Martin, 1991; Conforti, Kaibel, Walter & Weltge, 2015)**

Given a non-empty polyhedron $Q$ and $\gamma \in \mathbb{R}$, let

$$P := \{ x : \langle y, x \rangle \geq \gamma \text{ for all } y \in Q \}.$$

Then $xc(P) \leq xc(Q) + 1$.

**Proof:**

- Let $Q = \{ Tz : Az \leq b \}$, where $A$ has $m = xc(Q)$ rows.
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- Let $Q = \{ Tz : Az \leq b \}$, where $A$ has $m = xc(Q)$ rows.

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- Thus, $P = \{ x : \exists \lambda \leq \emptyset : A^T \lambda = T^T x, \langle b, \lambda \rangle \geq \gamma \}$ is an extension with $m + 1$ inequalities.
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Proof:
- Let \( Q = \{ Tz : Az \leq b \} \), where \( A \) has \( m = xc(Q) \) rows.

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\hat{x} \in P \iff \min \left\{ \langle \hat{x}, y \rangle : y \in Q \right\} \geq \gamma
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Consequences:
- \( xc(B(P)) \) and \( xc(P) \) differ by at most \( d \).
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  $$ \iff \min \{ \langle \hat{x}, Tz \rangle : Az \leq b \} \geq \gamma $$
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- Thus, $P = \{ x : \exists \lambda \leq 0 : A^T \lambda = T^T x, \langle b, \lambda \rangle \geq \gamma \}$ is an extension with $m + 1$ inequalities.

Consequences:

- $xc(B(P))$ and $xc(P)$ differ by at most $d$.
- $2^{\Omega(|T|)} \leq xc(P_{T-cut}(n)) \leq O(n^2 \cdot 2^{|T|})$. 
Polar object of radial cone:

- Any $v \in P$ defines a face $F_{B(P)}(v) := \{ y \in B(P) : \langle v, y \rangle = 1 \}$ of $B(P)$.
Polar object of radial cone:

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**Lemma**

Let \( P \subseteq \mathbb{R}^d_+ \) be a blocking polyhedron and let \( v \in P \).

(i) \( F_{B(P)}(v) = \{ y \in \mathbb{R}^d : \langle v, y \rangle = 1, \langle x, y \rangle \geq 1 \ \forall x \in K_P(v) \} \).

(ii) \( K_P(v) = \{ x \in \mathbb{R}^d : \langle y, x \rangle \geq 1 \ \forall y \in F_{B(P)}(v) \} \).

**Proof:**

- Elementary convex geometry
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**Proof:**
- Elementary convex geometry

**Consequence:**
- $xc(K_P(v))$ and $xc(F_{B(P)}(v))$ differ by at most 1.
- To prove lower or upper bounds on $xc(K_P(v))$, it suffices to do analyze the face $F_{B(P)}(v)$ of $B(P)$. 
Theorem (Ventura & Eisenbrand, 2003)

For every set $T \subseteq V_n$ with $|T|$ even and every vertex $v$ of $P_{T\text{-join}}(n)^\uparrow$, corresponding to a $T$-join $J \subseteq E_n$ in $K_n$, the extension complexity of the radial cone of $P_{T\text{-join}}(n)$ at $v$ is most $O(|J| \cdot n^2)$.
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Our new proof:
- By Lemma, theorem reduces to $xc(P)$ for $P := \left\{ x \in P_{T\text{-cut}}(n)^\uparrow : \sum_{e \in J} x_e = 1 \right\}$.
Radial Cones of $T$-Join Polyhedra

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- But $F_m$ is also a face of $P_{T'\text{-cut}}(n)^\uparrow$ for $T' = m$ (set containing the nodes).
Radial Cones of $T$-Join Polyhedra

**Theorem (Ventura & Eisenbrand, 2003)**

For every set $T \subseteq V_n$ with $|T|$ even and every vertex $v$ of $P_{T\text{-join}}(n)^\uparrow$, corresponding to a $T$-join $J \subseteq E_n$ in $K_n$, the extension complexity of the radial cone of $P_{T\text{-join}}(n)$ at $v$ is most $O(|J| \cdot n^2)$.

**Our new proof:**

- By Lemma, theorem reduces to $xc(P)$ for

$$P := \left\{ x \in P_{T\text{-cut}}(n)^\uparrow : \sum_{e \in J} x_e = 1 \right\}. $$

- For each $m \in J$, let $F_m$ be the face of $P$ with $x_m = 1$ (and $x_e = 0 \ \forall \ e \in J \setminus \{m\}$).
- But $F_m$ is also a face of $P_{T'\text{-cut}}(n)^\uparrow$ for $T' = m$ (set containing the nodes).
- We obtain $xc(F_m) \leq O(n^2 \cdot 2^{|T'|}) = O(n^2)$. 

Marcel D"{o}scher
Extended Formulations for Radial Cones

Radial Cones of $T$-Join Polyhedra

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Matthias Walter

Extended Formulations for Radial Cones

Magdeburg 2018
Theorem (Walter & Weltge, 2018)

For sets $T \subseteq V_n$ with $|T|$ even and vertices $v$ of $P_{T\text{-cut}}(n)^\dagger$, the extension complexity of the radial cone of $P_{T\text{-cut}}(n)$ at $v$ is least $2^{\Omega(|T|)}$. 

Proof:

Let $v = \chi(\delta(S))$.

Let $t_1 \in S$, $t_2 \in V_n \setminus S$ as well as $U_1 = S \setminus \{t_1\}$, $U_2 = (V_n \setminus (S \cup \{t_2\}))$.

Let $F$ be the face of $P$ with $x_{\{t_1, t_2\}} = 1$ and $x_e = 0$ for all edges between $U_1$, $U_2$ and $\{t_1, t_2\}$.

$F$ is a Cartesian product of a vector and two $(T \cap U_i)$-join polyhedra on $U_i$ for $i = 1, 2$, where $|T_1| + |T_2| = |T| - 2$.

We obtain $\text{xc}(P) \geq \text{xc}(F) \geq 2^{\Omega(|T|)}$ for $i = 1, 2$. 

Matthias Walter
Extended Formulations for Radial Cones
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Radial Cones of $T$-Cut Polyhedra

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Radial Cones of $T$-Cut Polyhedra

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For sets \( T \subseteq V_n \) with \( |T| \) even and vertices \( v \) of \( P_{T\text{-cut}}(n)^\uparrow \), the extension complexity of the radial cone of \( P_{T\text{-cut}}(n) \) at \( v \) is least \( 2^{\Omega(|T|)} \).

Proof:

- Let \( v = \chi(\delta(S)) \).
- By Lemma, theorem reduces to \( xc(P) \) for

\[
P := \left\{ x \in P_{T\text{-join}}(n)^\uparrow : \sum_{e \in \delta(S)} x_e = 1 \right\}.
\]

- Let \( t_1 \in S, t_2 \in V_n \setminus S \) as well as \( U_1 := S \setminus \{t_1\}, U_2 := (V_n \setminus (S \cup \{t_2\})) \).
- Let \( F \) be the face of \( P \) with \( x\{t_1,t_2\} = 1 \) and \( x_e = 0 \) for all edges between \( U_1, U_2 \) and \( \{t_1, t_2\} \).
- \( F \) ist a Cartesian product of a vector and two \( (T \cap U_i)\)-join polyhedra on \( U_i \) for \( i = 1, 2 \), where \( |T_1| + |T_2| = |T| - 2 \).
- We obtain \( xc(P) \geq xc(F) \geq 2^{\Omega(|T_i|)} \) for \( i = 1, 2 \).
Thanks!

Conclusion:

- Extended formulations can help, but only sometimes.
- Although polynomially solvable, there is no obvious way to solve the minimum-weight $T$-cut problem with LP techniques.
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Conclusion:

- Extended formulations can help, but only sometimes.
- Although polynomially solvable, there is no obvious way to solve the minimum-weight $T$-cut problem with LP techniques.

Other candidates for investigation:

- Stable-set polytopes of claw-free graphs (current work with Gianpaolo Oriolo and Gautier Stauffer).
- Stable-set polytopes of perfect graphs (polyhedral description is known, but best extended formulation has $O(n \log n)$ facets).