Solving Bulk-Robust Assignment Problems to Optimality

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Joint work with

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Viktor Bindewald & Dennis Michaels (TU Dortmund)

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Operations Research | RWTH AACHEN UNIVERSITY

Bulk-Robustness for Assignment Problems

Assignment Problem:

- Input: Bipartite graph $G = (V, E)$ with $V = A \mathbin{\dot\cup} B$, edge costs $c \in \mathbb{R}^E$
- Feasible sets: Perfect matchings $M \subseteq E$ (assuming $|A| = |B|$)
- Goal: Minimize cost $c(M) := \sum_{e \in M} c_e$

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Literature:

- Concept formally introduced by Adjiashvili, Stiller & Zenklusen (MPA 2015)
- Classical related problems: k-edge connected spanning subgraph problem robustifies spanning-tree problem against failure of any $(k-1)$-edge set.
- LP-based $\mathcal{O}(\log(|V|))$-approximation algorithm by Adjiashvili, Bindewald & Michaels (ICALP 2016)

Bulk-Robust Assignments with Edge Failures

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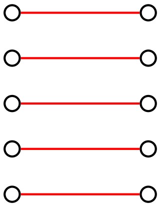

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- For any $d < 1$, it admits no $(d \log |V|)$-approximation, unless $\text{NP} \subseteq \text{DTIME}(|V|^{\log \log |V|})$.

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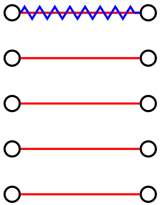

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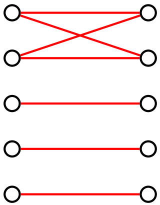

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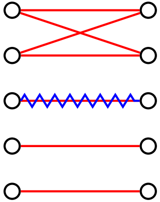

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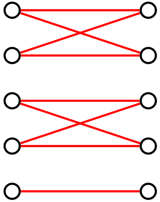

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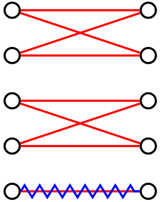

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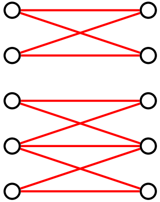

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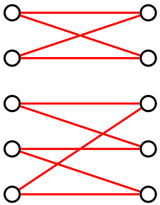

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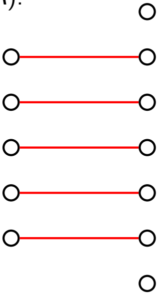

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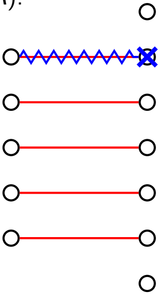

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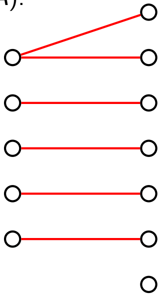

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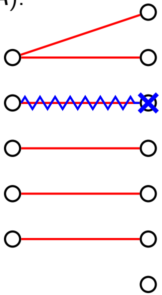

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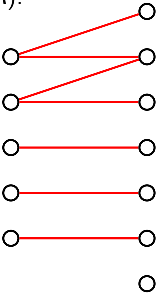

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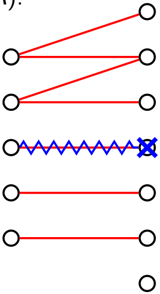

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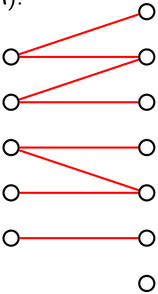

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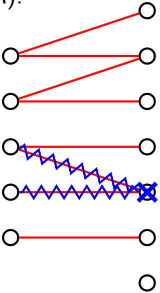

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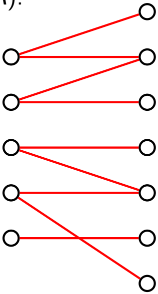

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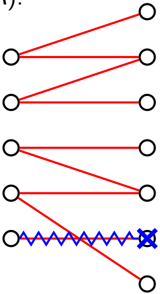

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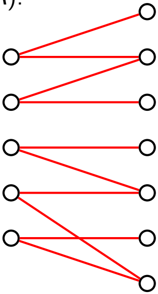

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General Case

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- Failure scenarios $\mathcal{F} = \{F_1, \ldots, F_\ell\}$ with $F_i \subseteq E$
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Special Cases:

- Edge failures: Set $k(F_i) := |A| = |B|$ and $F_i := \{f_i\}$ for all $i \in [\ell]$.
- Node failures: Set $k(F_i) := |A|$ and $F_i := \delta(b_i)$ for all $i \in [\ell]$.

Integer Programming Models

Straight-forward model (see Adjiashvili et al., ICALP 2016):

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\min\ \ & c^\top x \\
\text{s.t.}\ \ & x \geq y^{(F)} && \text{for all } F \in \mathcal{F} && (1)\\
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- Has $\mathcal{O}(|\mathcal{F}| \cdot |E|)$ variables and constraints.

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- Projection onto x is the dominant of the $k(F)$-matching polytope.

Integer Programming Models

Straight-forward model (see Adjiashvili et al., ICALP 2016):

$$\begin{aligned} \min\ & c^\top x & & \\ \text{s.t.}\ & x \geq y^{(F)} & \text{for all } F \in \mathcal{F} & \quad (1)\\ & y^{(F)} \in P_{k(F)\text{-match}}(G-F) & \text{for all } F \in \mathcal{F} & \quad (2)\\ & x_e \in \mathbb{Z}_+ & \text{for all } e \in E & \quad (3) \end{aligned}$$

- Has $\mathcal{O}(|\mathcal{F}| \cdot |E|)$ variables and constraints.

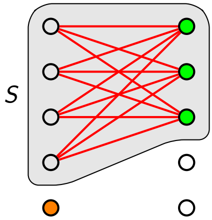
Polyhedral combinatorics helps:

- What does this mean for x?

$$\exists y : x \geq y,\ y \in P_{k(F)\text{-match}}(G')$$

- Projection onto x is the dominant of the $k(F)$-matching polytope.
- Inequalities known (Fulkerson 1970):

$$\sum_{e \in E[S]} x_e \geq |S| - |V| + k(F) \quad \text{for all } S \subseteq V$$



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\end{aligned}
$$

- Has $\mathcal{O}(|\mathcal{F}| \cdot |E|)$ variables and constraints.

Equivalent (derived from dominant):

$$
\begin{aligned}
\min \quad & c^\top x \\
\text{s.t.} \quad & \sum_{e \in E[S] \setminus F} x_e \geq |S| - |V| + k(F) && \text{for all } S \subseteq V \text{ for all } F \in \mathcal{F} && (4)\\
& x_e \in \mathbb{Z}_+ && \text{for all } e \in E && (5)
\end{aligned}
$$

- Has $\mathcal{O}(|E|)$ variables and $\mathcal{O}\left(|\mathcal{F}| \cdot 2^{|V|}\right)$ constraints.
- For every $F \in \mathcal{F}$, separation problem reduces to a minimum s-t-cut problem.

Computational Setup

Setup:

- System: KUbuntu on 2.6 GHz with 16 GB RAM
- Software: SCIPOptSuite 5.0.0 and LEMON Graph Library

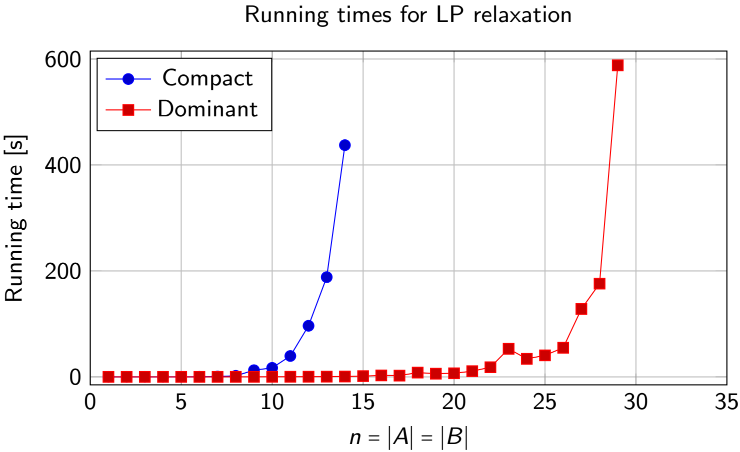
Parameters:

- General purpose cuts off
- Heuristics off in root node
- Time limit of 600 s
- Different further settings depending on experiment.

Models in Practice: LP Relaxation

Setup:

- Complete bipartite graphs with $|A| = |B| = n$
- Uniform failures $\mathcal{F} = \{\{e\} \mid e \in E\}$, unit costs $c = \mathbb{1}$
- Settings: only root node



Models in Practice: LP Relaxation

Setup:

- Erdős-Rényi graphs with $|A| = |B| = n$, $p = 0.5$
- Uniform failures $\mathcal{F} = \{\{e\} \mid e \in E\}$, unit costs $c = \mathbb{1}$
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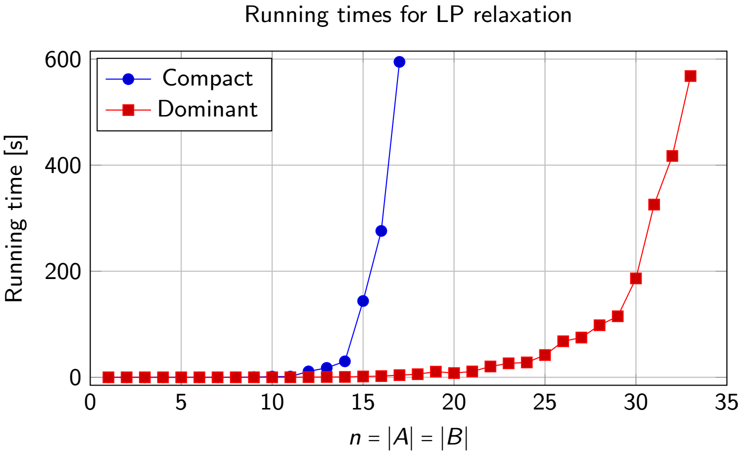

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Results for compact vs. dominant model (IP)

		Compact model			Dominant model		
n	**Opt**	**Root time**	**Final bnd**	**Time**	**Root time**	**Final bnd**	**Time**
5	10	0.0	10.0	6.7	0.0	10	0.2
6	12	0.1	12.0	252.5	0.0	12	0.2
7	14	0.4	10.5	600.0	0.0	14	0.3
8	16	1.7	10.8	600.0	0.0	16	22.6
9	18	5.8	11.1	600.0	0.0	18	0.4
10	20	14.8	11.4	600.0	0.0	20	243.2
11	22	41.2	12.2	600.0	0.0	14.8	600.0

Solving with Gurobi

Setup:

- Combination of complete graph, singleton failures and unit costs has lots of symmetry.
- Gurobi detects this and can prove lower bound earlier.
- For $n = 8$, we observe the following behavior:

Results for compact model with Gurobi (IP)

```
    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0    9.14286    0 3648   64.00000    9.14286  85.7%     -    4s
     0     0    9.14286    0 3648   64.00000    9.14286  85.7%     -    6s
     0     2    9.14286    0 3648   64.00000    9.14286  85.7%     -   13s
     2     2   10.00000    1 3521   64.00000   10.00000  84.4%  5738   15s
...
   525   355   13.00000   16 1267   16.00000   11.73810  26.6%  1526  205s
   542   349 infeasible   16        16.00000   13.42857  16.1%  1509  210s
   548   352   13.80000   19 2011   16.00000   13.80000  13.7%  1519  215s
   559   345   15.00000   19 1923   16.00000   14.00000  12.5%  1525  220s

Explored 568 nodes (899200 simplex iterations) in 223.95 seconds
Thread count was 1 (of 4 available processors)

Optimal solution found (tolerance 1.00e-04)
Best objective 1.600000000000e+01, best bound 1.600000000000e+01, gap 0.0%
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Note: This effect vanishes as soon as the graph is not symmetric anymore.

Strengthening the Model

Chvátal-Gomory cuts:

- Consider $F_1, \dots, F_\ell$ with constant $k(F_i) = k$ for all $i \in [\ell]$ ($\ell \geq 2$).
- Sum up all inequalities for fixed S with $|S| - |V| + k \geq 1$.

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$$\sum_{e \in E[S]} \left\{ \begin{array}{ll} 2 & \text{if } e \text{ in no } F_i \\ 0 & \text{if } e \text{ in all } F_i \\ 1 & \text{otherwise} \end{array} \right\} x_e \geq |S| - |V| + k + 1$$

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- Weakened for coefficients with e in no F_i.
- Strengthened for coefficients with e in all F_i.
- Stronger right-hand side.

Separation Problem

Input:

- Bipartite graph $G = (V, E)$ with bipartition $V = A \cup B$.
- Edge weights $w \in \mathbb{R}_+^E$
- Parameter k.

Goal:

- Find $S \subseteq V$ with $|S| \geq |V| - k + 1$ minimizing $w(E[S]) - |S| + |V| - k$

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IP Model:

- Variables y and z with
- $y_v = 1 \iff v \in S$
- $z_e = 1 \iff e \in E[S]$

$$\begin{aligned} \min \quad & -\sum_{v \in V} y_v + \sum_{e \in E} w_e z_e \\ \text{s.t.} \quad & -y_a - y_b + z_{a,b} \geq -1 \qquad \text{for all } \{a, b\} \in E \\ & y(A) + y(B) \geq |V| - k + 1 \\ & y, \; z \text{ binary} \end{aligned}$$

Observe: TU system plus a single inequality.

Bad News: NP-hardness

Separation problem:

- Input: bipartite graph $G = (V, E)$, a nonnegative vector $w \in \mathbb{Q}_+^E$ and a number $\ell \in \mathbb{N}$.
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Some NP-hard problem:

- Input: bipartite Graph $G = (V, E)$, numbers $m, n \in \mathbb{N}$.
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- Hardness: Apollonio & Simeone (2014)

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Reduction idea:

- Node complementing ($\ell := |V| - n$) and proper scaling ($w := (|V| + 1)\mathbb{1}_E$)
- Existence of S with $|S| \leq n$ and $|\{e \in E \mid e \cap S \neq \emptyset\}| \geq m$ is equivalent to existence of $\bar{S}$ with $|\bar{S}| \geq \ell$ and

$$\begin{aligned}
|E \setminus E[\bar{S}]| \geq m &\iff |E[\bar{S}]| \leq (|E| - m) \\
&\iff (|V| + 1)|E[\bar{S}]| \leq (|V| + 1)(|E| - m) \\
&\iff (|V| + 1)|E[\bar{S}]| - |\bar{S}| \leq (|V| + 1)(|E| - m) \\
&\iff w(E[\bar{S}]) - |\bar{S}| \leq (|V| + 1)(|E| - m).
\end{aligned}$$

(note that $0 \leq |\bar{S}| < |V| + 1$)

Good News: Nice Heuristic Approach

Main idea:

- Let's move $y(A) + y(B) \geq |V| - k + 1$ into the objective function!
- Lagrange multiplier is one-dimensional: (binary) search for good values.
- Subproblem again reduces to mininum s-t-cut problem.
- If it returns a set S then we have a most-violated inequality among all inequalities with this $|S|$.

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Desirable side-effect:

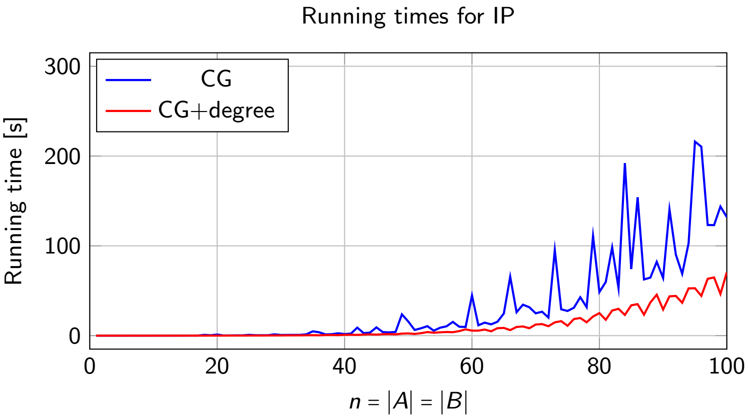
$$\sum_{e \in E[S]} \{0,1,2\}\, x_e \geq |S| - |V| + k + 1$$

- Chvátal-Gomory strengthening is stronger for small right-hand sides.
- We can control $|S|$ via Lagrange multipliers to get a small right-hand side.
- Experimentally best strategy: aim for violated cuts with minimum $|S|$.

Models in Practice: CG Cuts

Setup:

- Complete bipartite graphs with $|A| = |B| = n$
- Uniform failures $\mathcal{F} = \{\{e\} \mid e \in E\}$, unit costs $c = \mathbb{1}$



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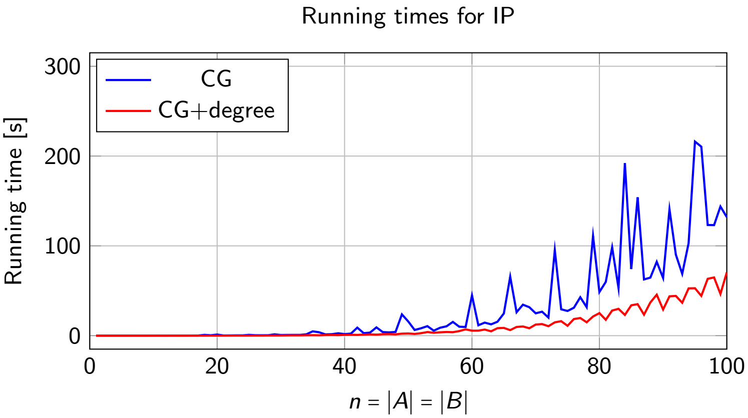
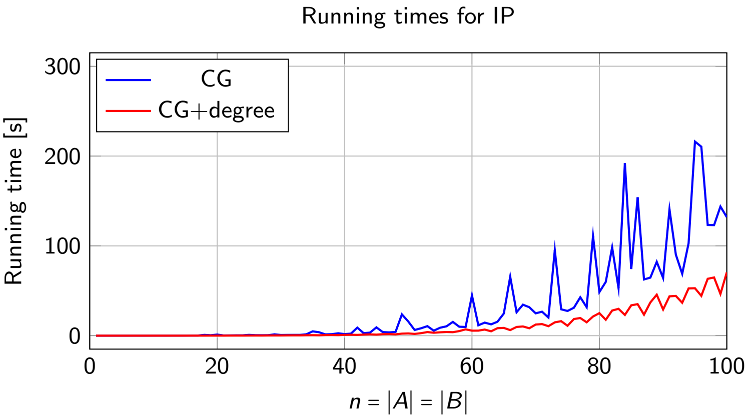


- Note that we are solving the **IP** and not just the relaxation!

Models in Practice: CG Cuts

Setup:

- Complete bipartite graphs with $|A| = |B| = n$
- Uniform failures $\mathcal{F} = \{\{e\} \mid e \in E\}$, unit costs $c = \mathbb{1}$
- Special case of CG cuts are strengthened degree inequalities $x(\delta(v)) \geq 2$.



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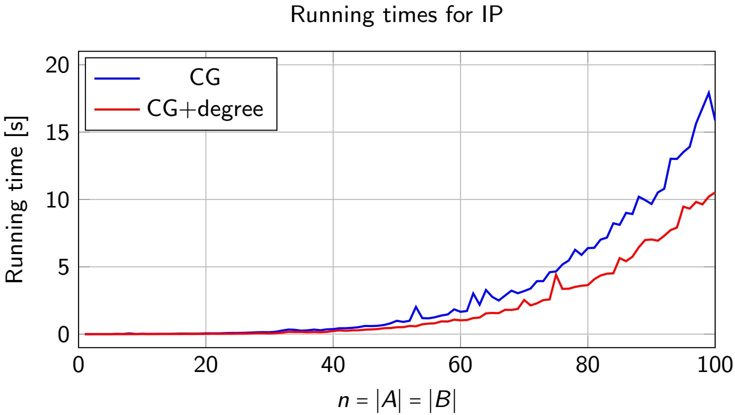

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Setup:

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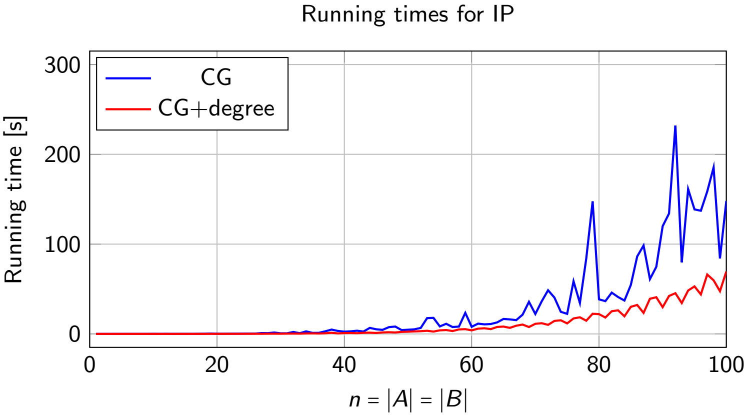
- Random costs $c_e \in \{1, \dots, 2\}$ for all $e \in E$ independently.



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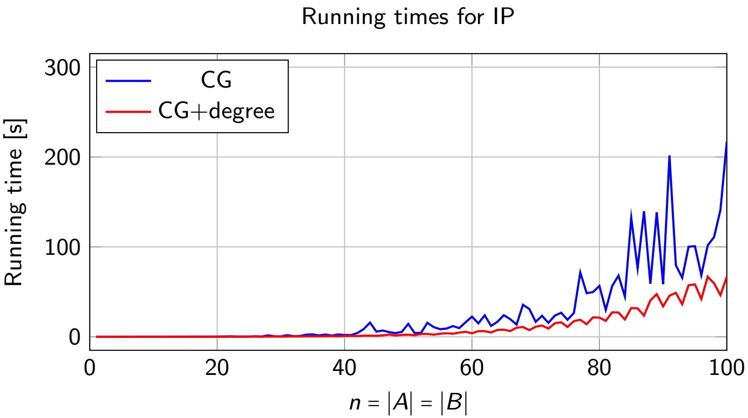
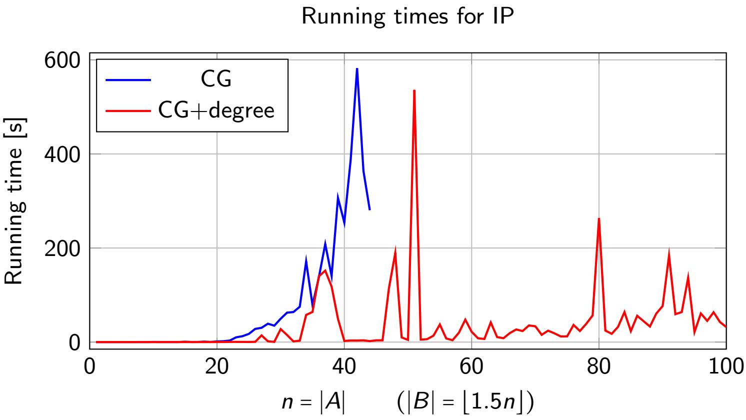


Models in Practice: IP Bounds with CG Cuts

Setup:

- Complete bipartite graphs with $|A| = n$ and $|B| = \lfloor 1.5n \rfloor$
- Node failures $\mathcal{F} = \{\delta(b) \mid b \in B\}$, unit costs $c = \mathbb{1}$



Remark: Problem is on primal side, i.e., finding an optimal solution!

Thanks!

Things you've seen:

- Speed-up of dominant formulation vs. compact one.
- Derivation of Chvátal-Gomory (CG) cuts.
- Fast heuristic separation with Lagrange multiplier.
- Strength of CG cuts, in particular strengthened degree.

Thanks!

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- Fast heuristic separation with Lagrange multiplier.
- Strength of CG cuts, in particular strengthened degree.

Things you might see in the future:

- Generalization to arbitrary s-t-flows (thanks to Britta Peis).
- Structured instances:
  - ...obtained from the SetCover reduction
  - ...obtained from other sources (QAPLIB?)
  - ...yours?
- Implementation of / comparison with approximation algorithm