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The strength of Dantzig–Wolfe reformulations for the stable set and related problems

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ABSTRACT

Dantzig–Wolfe reformulation of an integer program convexifies a subset of the constraints, which yields an extended formulation with a potentially stronger linear programming (LP) relaxation. We would like to better understand the strength of such reformulations in general. As a first step we investigate the classical edge formulation for the stable set problem. We characterize weakest possible Dantzig–Wolfe reformulations (with LP relaxations not stronger than the edge formulation) and strongest possible reformulations (yielding the integer hull). We (partially) extend our findings to related problems such as the matching problem and the set packing problem. These are the first non-trivial general results about the strength of relaxations obtained from decomposition methods, after Geoffrion's seminal 1974 paper about Lagrangian relaxation.

1. Introduction

Consider the integer program

$$
\begin{array}{lll}
(IP) & \max & c^T x \\
 & \text{s.t.} & a_i^T x \leq b_i \quad \forall i \in I \\
 & & x \in \mathbb{Z}^n \cap [\ell, u],
\end{array}
$$

where I denotes a finite index set, $n \in \mathbb{Z}_{>0}$, $\ell, u, c, a_i \in \mathbb{Q}^n$ and $b_i \in \mathbb{Q}$ for $i \in I$. The *integer hull* P_{IP} of (IP) is defined as

$$P_{IP} := \operatorname{conv}\left\{x \in \mathbb{Z}^n \cap [\ell, u] : a_i^T x \leq b_i \ \forall i \in I\right\}.$$

The *fractional polytope* P_{LP} contains all solutions that are feasible to the linear programming (LP) relaxation of (IP), i.e.,

$$P_{LP} := \left\{x \in \mathbb{R}^n \cap [\ell, u] : a_i^T x \leq b_i \ \forall i \in I\right\}.$$

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Let $I' \subseteq I$ and define the integer hull corresponding to the integer program that only consists of constraints with index in I' as

$$X(I') := \operatorname{conv}\left\{x \in \mathbb{Z}^n \cap [\ell, u] : a_i^T x \leq b_i \ \forall i \in I'\right\}.$$

In Dantzig–Wolfe reformulation for integer programs every solution $x \in X(I')$ is reformulated as a convex combination of extreme points of $X(I')$. Thereby the variables x are replaced by new variables, one for each extreme point of $X(I')$, determining the coefficients in the convex combination. When we solve the LP relaxation of the resulting integer program, we implicitly optimize over the polytope

$$P_{DW}(I') := \left\{x \in \mathbb{R}^n \cap [\ell, u] : \ a_i^T x \leq b_i \ \forall i \in I \setminus I', x \in X(I')\right\},$$

which corresponds to *convexifying* the constraints with index in I' [1]. We remark that the polytope $P_{DW}(I')$ is also obtained when the constraints with index in $I \setminus I'$ are dualized in Lagrangian relaxation [2] or when all valid inequalities for $X(I')$ are added to the LP relaxation of (IP). Furthermore, note that optimizing over $P_{DW}(I')$ can be done in polynomial time if optimizing over $X(I')$ can be done in polynomial time.

The strength of relaxations of integer programs is a classical and well-studied topic in polyhedral combinatorics. For Dantzig–Wolfe reformulations in general we obviously have that

$$P_{IP} \subseteq P_{DW}(I') \subseteq P_{LP} \quad \forall I' \subseteq I. \tag{1}$$

Since we are completely free to choose $I' \subseteq I$, and in particular both extreme cases $P_{IP} = P_{DW}(I)$ and $P_{LP} = P_{DW}(\emptyset)$ are possible, we must necessarily relate the notion of strength to the set of convexified constraints. It was proven in the context of Lagrangian relaxation by Geoffrion [2] that the relation $P_{DW}(I') \subsetneq P_{LP}$ holds only if

$$X(I') \subsetneq \left\{x \in \mathbb{R}^n \cap [\ell, u] : a_i^T x \leq b_i \ \forall i \in I'\right\}. \tag{2}$$

Notice that the opposite direction is not true in general: If Eq. (2) holds, but all valid inequalities for $X(I')$ are dominated by constraints $a_i^T x \leq b_i$ for $i \in I$, then $P_{LP} \subseteq X(I')$, and hence, $P_{DW}(I') = P_{LP}$. Interestingly, this is already all we know about the strength of Dantzig–Wolfe reformulations of integer programs in general. In order to make further progress, we will now focus on a particular problem but stay generic in terms of reformulations.

In this context, the stable set problem is a canonical problem to study. In fact, the stable set problem was used to understand the strength of other types of relaxations such as the Sherali–Adams, Lovász–Schrijver, and Lasserre relaxations [3], the corner relaxation [4–6], as well as some other families of relaxations [7]. Note also that the stable set problem, and the closely related more general set packing problem, appear in more general integer programs to model all kinds of “conflicts” arising in many applications. In fact, such structures are extracted from general integer programs in the form of the conflict graph [8] in which feasible solutions to the integer program correspond to stable sets. This in turn can be exploited when separating cutting planes (such as clique or odd cycle inequalities).

The literature knows several Dantzig–Wolfe reformulations of the classical textbook model for the stable set problem. This model has a constraint for each edge (see below), so that we interchangeably speak of convexifying a subset of constraints and convexifying a subgraph. Warrier et al. [9] presented a branch-and-price approach for the stable set problem, where they apply Dantzig–Wolfe reformulation either by convexifying chordal induced subgraphs (such that optimizing over $P_{DW}(I')$ can be done in polynomial time) or by convexifying induced subgraphs using a heuristic partitioning of the node set of the graph. Sachdeva [10] extended the idea of Warrier et al. by using a partition of the edge set created from a partition of the node set to obtain a potentially stronger Dantzig–Wolfe reformulation. Ribeiro et al. [11] used a similar idea when they applied Lagrangian relaxation to stable set problems arising in map labeling problems. Campêlo and Corrêa [12] took a completely different approach and presented a Lagrangian relaxation based on representatives of stable sets. Another idea of Warrier et al. was picked up by Gabrel [13], who convexified perfect subgraphs such that optimizing over $P_{DW}(I')$ can again be done in polynomial time.

In contrast to all previous rather empirical work, that was also limited to reformulating specific classes of subgraphs, we are interested in more fundamental and more general results. We stress that at this time, our work is not meant to imply any computationally viable solution procedure for the stable set problem. Our main results are a characterization of Dantzig–Wolfe reformulations for the stable set problem with $P_{LP} = P_{DW}(I')$ as well as a characterization of Dantzig–Wolfe reformulations with $P_{IP} = P_{DW}(I')$. These findings give new insights on the stable set polytope and (partially) extend to related problems like the matching and the set packing problem. To the best of our knowledge, after Geoffrion's 1974 paper [2], these are the first non-trivial general results about the strength of relaxations obtained from decomposition methods.

2. The stable set problem

Let $G = (V, E)$ be a simple, undirected graph with $n := |V|$ nodes. A set of nodes $S \subseteq V$ is called a *stable set* in G if no nodes of S are adjacent in G, i.e., $|e \cap S| \leq 1$ for all $e \in E$. Note that each edge $e \in E$ is a set of nodes $e = \{u, v\}$ for some $u, v \in V$ with $u \neq v$. Abbreviating, we denote the edge $\{u, v\}$ with uv. Let $\mathcal{S}(G)$ be the set of all stable sets in G. A stable set S^* is called *maximum stable set* if $S^* \in \operatorname{argmax}\{|S| : S \in \mathcal{S}(G)\}$. The maximum cardinality $\alpha(G) := \max\{|S| : S \in \mathcal{S}(G)\}$ is called *stability number* of G. Given a weighting $w \in \mathbb{Z}^V_{\geq 0}$ on the nodes, a stable set S^* in G with weighting w is called *maximum weighted stable set* if $S^* \in \operatorname{argmax}\{\sum_{v \in S} w_v : S \in \mathcal{S}(G)\}$. The maximum weight $\alpha_w(G) := \max\{\sum_{v \in S} w_v : S \in \mathcal{S}(G)\}$ is called *weighted stability number* of G with weighting w. We denote the weight of stable set S using weighting w with $w(S) := \sum_{v \in S} w_v$.

The classical way of formulating the maximum weighted stable set problem as an integer linear program is to introduce a binary variable $x_v \in \{0, 1\}$ for each node $v \in V$ indicating whether node v is in the stable set ($x_v = 1$) or not ($x_v = 0$). Furthermore, we have a constraint for each edge $e \in E$ enforcing that at most one node in the stable set is incident to edge e. The objective is to maximize the weight of the stable set. This leads to the following so-called *edge formulation*:

$$\begin{aligned} \max \quad & \sum_{v \in V} w_v \cdot x_v \\ \text{s.t.} \quad & x_u + x_v \leq 1 \qquad \forall uv \in E \\ & x \in \{0, 1\}^V. \end{aligned}$$

For convenience, we denote by $\{0, 1\}^V$ the set of n-dimensional binary vectors, where the dimensions are labeled according to the elements in V. The *stable set polytope* $\mathrm{STAB}(G)$ is defined as the convex hull of incidence vectors of stable sets in G, i.e.,

$$\mathrm{STAB}(G) := \operatorname{conv}\{x \in \{0, 1\}^V : x_u + x_v \leq 1 \ \forall uv \in E\}.$$

The set of LP-feasible solutions for the edge formulation is denoted by $\mathrm{FRAC}(G)$ and is defined as

$$\mathrm{FRAC}(G) := \{x \in [0, 1]^V : x_u + x_v \leq 1 \ \forall uv \in E\}.$$

We refer to $\mathrm{FRAC}(G)$ as the *fractional stable set polytope* and we call a solution $\bar{x} \in \mathrm{FRAC}(G)$ a *fractional stable set* in G.

In Dantzig–Wolfe reformulation for integer programs a subset of the constraints is implicitly *convexified*. Let $E' \subseteq E$ be a subset of the edges of G and define $G' := (V, E')$. We will convexify all constraints corresponding to edges in E':

$$\mathrm{DW}(G, G') := \{x \in [0, 1]^V : x_u + x_v \leq 1 \ \forall uv \in E \setminus E', \ x \in \mathrm{STAB}(G')\}.$$

Specializing (1), the previously defined polytopes relate as

$$\mathrm{STAB}(G) \subseteq \mathrm{DW}(G, G') \subseteq \mathrm{FRAC}(G).$$

We want to investigate the strength of Dantzig–Wolfe reformulations by investigating the polytope $\mathrm{DW}(G, G')$. Especially, we are interested in conditions for G' such that either $\mathrm{STAB}(G) = \mathrm{DW}(G, G')$ or $\mathrm{DW}(G, G') = \mathrm{FRAC}(G)$ holds. Obviously, the identity $\mathrm{STAB}(G) = \mathrm{DW}(G, G)$ holds because every constraint is convexified. In this case the reformulation is *strongest possible*. If we choose $E' = \emptyset$, the identity $\mathrm{FRAC}(G) = \mathrm{DW}(G, G')$ holds and the reformulation is *weakest possible*. In the next section, we characterize when exactly these strongest and weakest Dantzig–Wolfe reformulations for the stable set problem occur.

Notation. Let $G = (V, E)$ be a graph. For $V' \subseteq V$ we denote by $G[V']$ the subgraph induced by V'. We use $E(G')$ to refer to the edge set of any subgraph $G' = (V', E')$ of G, i.e., $E(G') = E'$. Analogously define a subset of nodes $V(G') = V'$. Let further $G - C := (V, E \setminus C)$ for any $C \subseteq E$ denote the graph obtained from G by deleting all edges $e \in C$. We analogously define $G + C$. For $V' \subseteq V$, $N(V') := \{v \in V \setminus V' \mid u \in V', uv \in E\}$ defines the set of neighbors of V'.

3. Weakest possible Dantzig–Wolfe reformulations

Nemhauser and Trotter [14] characterized graphs G such that the stable set polytope $\mathrm{STAB}(G)$ equals the fractional stable set polytope $\mathrm{FRAC}(G)$.

Theorem 1 ([14]). *Let $G = (V, E)$ be a graph. Then $\mathrm{FRAC}(G) = \mathrm{STAB}(G)$ holds if and only if G is bipartite.*

Thus, if G is bipartite, all three polytopes $\mathrm{FRAC}(G)$, $\mathrm{DW}(G, G')$, and $\mathrm{STAB}(G)$ coincide, no matter how we choose the graph G'. Note that the graph G' is always bipartite in this case.

A graph is bipartite if and only if it does not contain an odd cycle, i.e., a cycle with an odd number of nodes (and an odd number of edges). Let $C = (V_C, E_C)$ be an odd cycle in G with $|V_C| = 2k + 1$ for some $k \in \mathbb{Z}_{>0}$. The following *odd cycle inequality* [15] is valid for the stable set polytopes $\mathrm{STAB}(G[V_C])$ and $\mathrm{STAB}(G)$:

$$\sum_{v \in V_C} x_v \leq k.$$

We will use Theorem 1 and odd cycle inequalities in order to prove the following characterization of weakest possible Dantzig–Wolfe reformulations:

Theorem 2. *Let $G = (V, E)$ be a graph, let $E' \subseteq E$ be a subset of the edges, and define $G' := (V, E')$. Then $\mathrm{DW}(G, G') = \mathrm{FRAC}(G)$ if and only if G' is bipartite.*

Proof. Consider the definitions of the polytopes:

$$\begin{aligned}
\mathrm{DW}(G, G') &= \{x \in [0,1]^V : x_u + x_v \leq 1 \ \forall uv \in E \setminus E',\ x \in \mathrm{STAB}(G')\}, \\
\mathrm{FRAC}(G) &= \{x \in [0,1]^V : x_u + x_v \leq 1 \ \forall uv \in E\} \\
&= \{x \in [0,1]^V : x_u + x_v \leq 1 \ \forall uv \in E \setminus E',\ x \in \mathrm{FRAC}(G')\}.
\end{aligned}$$

It is easy to see that $\mathrm{STAB}(G') = \mathrm{FRAC}(G')$ implies $\mathrm{DW}(G, G') = \mathrm{FRAC}(G)$. This also follows directly from Geoffrion's result [2].

Now suppose $\mathrm{DW}(G, G') = \mathrm{FRAC}(G)$ holds and assume that $\mathrm{STAB}(G') \neq \mathrm{FRAC}(G')$. Then there exists an odd cycle $C = (V_C, E_C)$ in G'. Let $\bar{x}$ be the solution with $\bar{x}_v = \frac{1}{2}$ for $v \in V_C$ and $\bar{x}_v = 0$ otherwise. The solution $\bar{x}$ is obviously in $\mathrm{FRAC}(G)$, but $\bar{x}$ is not in $\mathrm{STAB}(G')$, because it does not satisfy the odd cycle inequality corresponding to the odd cycle C. Since $\mathrm{STAB}(G') \supseteq \mathrm{DW}(G, G')$, this implies $\bar{x} \notin \mathrm{DW}(G, G')$ and therefore $\mathrm{DW}(G, G') \neq \mathrm{FRAC}(G)$, which is a contradiction to the assumption $\mathrm{DW}(G, G') = \mathrm{FRAC}(G)$.

Thus, the equation $\mathrm{DW}(G, G') = \mathrm{FRAC}(G)$ holds if and only if $\mathrm{STAB}(G') = \mathrm{FRAC}(G')$. Together with Theorem 1 this proves the theorem. □

Notice that Theorem 2 (in combination with Theorem 1) shows that Geoffrion's necessary condition [2] for $P_{DW}(I') \subsetneq P_{LP}$ is sufficient when considering the edge formulation for the stable set problem.

4. Strongest possible Dantzig–Wolfe reformulations

Theorem 2 states that a Dantzig–Wolfe reformulation is stronger than the LP relaxation if and only if G' is not bipartite. Hence, G' must contain some odd cycle. An *odd induced cycle* is an induced cycle with an odd number of nodes, that is, at least three. An edge e is called *chord* of a cycle $C = (V_C, E_C)$ in G if $e \notin E_C$, but $e \in E(G[V_C])$. Hence, an odd induced cycle is an odd cycle without chords. The literature calls odd induced cycles with at least five nodes *odd holes*. Since we explicitly include cycles on three nodes (triangles or 3-cliques) we stick to odd induced cycles to avoid confusion. Note that a graph contains an odd induced cycle if and only if it contains an odd cycle. Thus, a graph is bipartite if and only if it does not contain an odd induced cycle. We prove in this section that we obtain a strongest possible reformulation, i.e., the integer hull, if and only if G' contains *all* odd induced cycles.

4.1. Necessary condition

Theorem 3. *Let $G = (V, E)$ be a graph, let $E' \subseteq E$, and define $G' := (V, E')$. If $\mathrm{DW}(G, G') = \mathrm{STAB}(G)$ holds, then G' contains all odd induced cycles of G.*

Proof. Suppose that $\mathrm{DW}(G, G') = \mathrm{STAB}(G)$ holds and assume there exists an odd induced cycle $H = (V_H, E_H)$ that is not contained in G', i.e., $E_H \not\subseteq E'$. Hence, there exists an edge $e \in E_H$ with $e \notin E'$. Let $V_H = \{v_1, v_2, \ldots, v_{2k+1}\}$ and

$$E_H = \{v_i v_{i+1} : i = 1, \ldots, 2k\} \cup \{v_1 v_{2k+1}\}$$

for some $k \in \mathbb{Z}_{>0}$. Furthermore, let $e = v_1 v_{2k+1}$.

The solution $\bar{x}$ with $\bar{x}_v = \frac{1}{2}$ for $v \in V_H$ and $\bar{x}_v = 0$ otherwise is obviously not in $\mathrm{STAB}(G)$, because the odd cycle inequality $\sum_{v \in V_H} x_v \leq k$ is not satisfied. The solution $\bar{x}$ is a convex combination of incidence vectors x^{even} and x^{odd} of the stable sets $S_{even} := \{v_{2\ell} : \ell = 1, \ldots, k\}$ and $S_{odd} := \{v_{2\ell+1} : \ell = 0, \ldots, k\}$ in G', respectively, using coefficients $\frac{1}{2}$ for both incidence vectors, i.e., $\bar{x} = \frac{1}{2} x^{even} + \frac{1}{2} x^{odd}$. Thus, $\bar{x} \in \mathrm{STAB}(G')$ holds. Furthermore, the edge inequalities $x_u + x_v \leq 1 \ \forall uv \in E \setminus E'$ are satisfied, which implies that $\bar{x} \in \mathrm{DW}(G, G')$ holds. This contradicts the assumption $\mathrm{DW}(G, G') = \mathrm{STAB}(G)$. □

Note that Theorem 3 is not true if we replace odd induced cycles by odd cycles. The graph $G = (V, E)$ with $V = \{v_1, \ldots, v_5\}$ and $E = \{v_i v_{i+1} : 1 \leq i \leq 4\} \cup \{v_5 v_1, v_2 v_5\}$ gives a counter example. If we choose the edge set of the graph G' as $E' := \{v_1 v_2, v_2 v_5, v_5 v_1\}$, we already obtain $\mathrm{DW}(G, G') = \mathrm{STAB}(G)$, although the odd cycle $C = (V_C, E_C)$ with $V_C = V$ and $E_C = E \setminus \{v_2 v_5\}$ is not covered by G'.

4.2. Sufficient condition

On our way to prove the converse of Theorem 3 we analyze the structure of (weighted graphs that define) facets of the stable set polytope.

The inequalities $x_v \geq 0$ and $x_v \leq 1$ for $v \in V$ are called *trivial inequalities*. The inequalities $x_v \geq 0$ for $v \in V$ define facets of STAB(G), and the inequality $x_v \leq 1$ for some $v \in V$ is a facet of STAB(G) if and only if v is an isolated node. All other facets are called *non-trivial*. The edge inequalities $x_u + x_v \leq 1$ for $uv \in E$ define facets (called *edge-facets*) if and only if uv is a maximal clique [15]. Let $\sum_{v \in V} \pi_v x_v \leq \pi_0$ be a non-trivial, non-edge facet of STAB(G). Then $\pi_v \geq 0$ for all $v \in V$, $\pi_0 > 0$, and $\pi_0 = \alpha_\pi(G)$ holds [16–18]. Let $V_0 := \{v \in V : \pi_v > 0\}$ be the set of nodes corresponding to non-zero entries of π and let $G_0 := G[V_0] = (V_0, E_0)$ be the subgraph of G induced by V_0. Using a simple projection argument, it is easy to see that the inequality $\sum_{v \in V_0} \pi_v x_v \leq \alpha_\pi(G_0)$ defines a facet of STAB(G_0) [16]. This implies that the identity $\pi_0 = \alpha_\pi(G_0)$ holds. A graph $G_0 = (V_0, E_0)$ with weights $\pi_v > 0$ for $v \in V_0$ will be called *facet-graph* [16,17] if $\sum_{v \in V_0} \pi_v x_v \leq \pi_0$ defines a facet of STAB(G_0).

The following useful lemma gives us an idea of the special structure of facet-graphs.

Lemma 1 (*[18]*). *Let $G_0 = (V_0, E_0)$ with weights $\pi_v > 0$ for all $v \in V_0$ be a connected facet-graph with $|V_0| \geq 3$ and let $uv \in E_0$ be an edge. Then there exists a maximum weighted stable set in G_0 that contains neither u nor v.*

The following definition was also used in [16–18]. An edge $e \in E$ in a graph $G = (V, E)$ with weighting π is called *critical* if $\alpha_\pi(G - \{e\}) > \alpha_\pi(G)$ holds. A graph G with weighting π is called *α_π-critical* if every edge $e \in E$ is critical.

The class of α_π-critical graphs contains the well-known *α-critical* graphs, which are α_π-critical graphs with weights $\pi_v = 1$ for all $v \in V$. For α-critical graphs the following theorem was proven by Andrásfai [19] and inspired similar results [20,21] for α-critical graphs.

Theorem 4 (*[19]*). *Let $G_0 = (V_0, E_0)$ with weights $\pi_v = 1$ for all $v \in V_0$ be a connected graph with $|V_0| \geq 3$ and let $e \in E_0$ be a critical edge. Then there exists an odd induced cycle $H = (V_H, E_H)$ in G_0 containing the edge e, i.e., $e \in E_H$.*

The following result is a generalization of Theorem 4 to the weighted case using a similar proof idea as Andrásfai [19] used for the unweighted case.

Theorem 5. *Let $G_0 = (V_0, E_0)$ with weights $\pi_v > 0$ for all $v \in V_0$ be a connected facet-graph with $|V_0| \geq 3$ and let $e \in E_0$ be a critical edge. Then there exists an odd induced cycle $H = (V_H, E_H)$ in G_0 containing the edge e, i.e., $e \in E_H$.*

Proof. Let $e = uv \in E_0$ be a critical edge. Lemma 1 implies that there exists a maximum weighted stable set S in G_0 with $u, v \notin S$. Let S^+ be a maximum weighted stable set in $G_0^+ := G_0 - \{e\}$. Since we deleted a critical edge to obtain G_0^+ from G_0, the weight of a maximum weighted stable set in G_0^+ increases compared to a maximum weighted stable set in G_0. Hence, $\pi(S^+) > \pi(S)$ holds. Additionally, it holds that $u, v \in S^+$ because otherwise S^+ contained at most one of the nodes incident to the deleted edge $e = uv$ and, thus, S^+ would have been a stable set in G_0 with larger weight than S (which would be a contradiction to the maximality of S). Consider the induced bipartite subgraph $G_{bip} := G_0^+[S^+ \setminus S \cup S \setminus S^+]$ of G_0^+ with bipartition $(S^+ \setminus S, S \setminus S^+)$.

Assume that u and v are in different connected components of G_{bip}. First, we remark that there are no isolated nodes in G_{bip}, because otherwise we could increase the weight of either S or S^+ which would contradict the definition of these maximum weighted stable sets in G_0 and G_0^+, respectively. Hence, each connected component of G_{bip} has nodes in $S \setminus S^+$ as well as in $S^+ \setminus S$. Since

$$\pi(S^+ \setminus S) = \pi(S^+) - \pi(S^+ \cap S) > \pi(S) - \pi(S^+ \cap S) = \pi(S \setminus S^+)$$

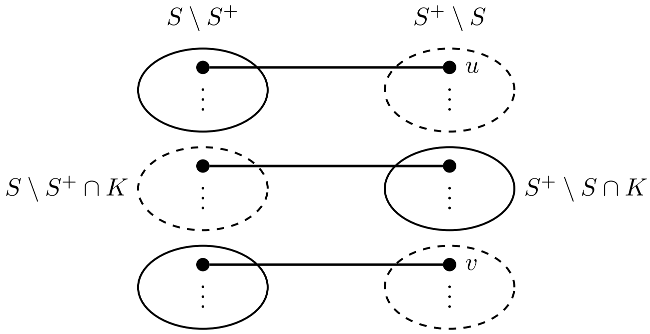

Fig. 1. A sketch of the connected components of the graph G_{bip} and the stable set S^{++} depicted with solid border (under the assumption that u and v are in different connected components of G_{bip}).

holds, there exists a connected component K of G_{bip} with

$$\pi(S^+ \setminus S \cap K) > \pi(S \setminus S^+ \cap K). \tag{3}$$

Note that there does not exist an edge between the node set $V(G_{bip}) = (S \cup S^+) \setminus (S \cap S^+)$ and the node set $S \cap S^+$ in G_0. Since the connected component of K is by definition not connected to the rest of the graph G_{bip} and K contains by assumption not both u and v, we can replace $(S \setminus S^+) \cap K$ by $(S^+ \setminus S) \cap K$ in the stable set S in order to obtain the following new stable set in G_0

$$S^{++} := (S \cap S^+) \cup ((S \setminus S^+) \setminus K) \cup (S^+ \setminus S \cap K) \tag{4}$$

as sketched in Fig. 1.

Using inequality (3) in combination with the definition (4) of S^{++}, we know that the weight $\pi(S^{++})$ of the stable set S^{++} in G_0 is greater than the weight $\pi(S)$ of the stable set S in G_0:

$$\pi(S^{++}) = \pi(S^{++} \cap S) + \pi(S^+ \setminus S \cap K) > \pi(S^{++} \cap S) + \pi(S \setminus S^+ \cap K) = \pi(S).$$

This contradicts the condition that S is a maximum weighted stable set in G_0. Hence, u and v are in the same connected component of G_{bip}.

Let $P = (u, w_1, \ldots, w_\ell, v)$ be some u-v-path of shortest length in G_{bip}. Note that P is an induced path in G_{bip}. Since u and v are contained in the same partite set, the path has even length, i.e., $\ell + 1 = 2k$ for some $k \in \mathbb{Z}_{>0}$. The odd cycle $H = (V_H, E_H)$ of G_0 with $V_H := uv \cup \{w_i : i = 1, \ldots, \ell\}$ and $E_H := \{uw_1, w_\ell v, uv\} \cup \{w_i w_{i+1} : i = 1, \ldots, \ell - 1\}$ is chordless and contains the edge e, which concludes the proof. □

Since not every edge of a facet-graph is critical, we will introduce a more general definition by considering subsets of edges. A subset $C \subsetneq E$ of the edges is called *critical* if $\alpha_\pi(G - C) > \alpha_\pi(G)$. The critical set C of edges is *minimal* if $\alpha_\pi(G - (C \setminus \{e\})) = \alpha_\pi(G)$ holds for all $e \in C$. Note that a set $C = \{e\}$ containing exactly one edge e is critical if and only if the edge e is critical.

The following lemma generalizes the result of Theorem 5 from critical edges to critical sets of edges.

Lemma 2. *Let $G_0 = (V_0, E_0)$ with weights $\pi_v > 0$ for all $v \in V_0$ be a connected facet-graph with $|V_0| \geq 3$ and let $C \subsetneq E_0$ be a minimal critical set of edges. Then there exists an edge $e \in C$ that is contained in an odd induced cycle $H = (V_H, E_H)$ in G_0, i.e., $e \in E_H$.*

Proof. Since C is a minimal critical set of edges, each edge $e \in C$ is critical in $G_0 - (C \setminus \{e\})$. Let $C = \{e_0, \ldots, e_{|C|-1}\}$ and $C_k := \{e_i : k \leq i \leq |C| - 1\} = \{e_k, \ldots, e_{|C|-1}\}$ for all $k \in \mathbb{Z}$ with $1 \leq k \leq |C|$. We will prove by induction on $k \in \mathbb{Z}$ with $1 \leq k \leq |C|$ that there exists an edge $e \in C \setminus C_k = \{e_0, \ldots, e_{k-1}\}$ that is contained in an odd induced cycle of $G_k := G_0 - C_k$.

In case $k = 1$, the edge e_0 is critical in $G_0 - (C \setminus \{e_0\})$ and by Theorem 5 contained in an odd induced cycle of $G_0 - (C \setminus \{e_0\}) = G_0 - C_1 = G_1$.

Suppose the claim holds for some fixed $k \in \mathbb{Z}$ with $1 \leq k \leq |C| - 1$ and let $e \in C \setminus C_k = \{e_0, \ldots, e_{k-1}\}$ be an edge that is contained in an odd induced cycle $H = (V_H, E_H)$ in G_k. We will prove that the claim also holds for $k + 1$. When adding edge e_k to the graph $G_k = G_0 - C_k$ in order to obtain $G_{k+1} = G_0 - C_{k+1}$, the following cases can occur:

1. $|e_k \cap V_H| \leq 1$: In this case H is also an odd induced cycle in G_{k+1} and the edge $e \in C \setminus C_k \subseteq C \setminus C_{k+1}$ is still an edge of the odd induced cycle H.
2. $|e_k \cap V_H| = 2$: In this case e_k is a chord of the odd cycle H in G_{k+1} and a new odd induced cycle H^+ in G_{k+1} with $e_k \in E(H^+)$ and $E(H^+) \subsetneq E_H \cup \{e_k\}$ arising when the edge e_k was added to the graph G_k.

By induction this proves the claim for $k \in \mathbb{Z}$ with $1 \leq k \leq |C|$. For $k = |C|$ we obtain the statement of the theorem. □

We have seen that odd induced cycles and (minimal) critical (sets of) edges in facet-graphs are strongly linked. The following lemma extends this link even more.

Lemma 3. *Let $G_0 = (V_0, E_0)$ with weights $\pi_v > 0$ for all $v \in V_0$ be a connected facet-graph with $|V_0| \geq 3$. Then there exists a connected α_π-critical subgraph $T = (V_0, E(T))$ with $\alpha_\pi(G_0) = \alpha_\pi(T)$ such that every edge $e \in E(T)$ is contained in some odd induced cycle H_e in G_0, i.e., $e \in E(H_e)$.*

Proof. A connected α_π-critical subgraph $T = (V_0, E(T))$ of G_0 can be obtained as follows [17,18]:

1. Let $i := 0$.
2. If G_i is α_π-critical, define $T := G_i$ as the α_π-critical subgraph, otherwise continue.
3. If there exists a non-critical edge of G_i that is *not* contained in any odd induced cycle in G_0, we select such an edge as e_i. Otherwise, let e_i be any arbitrary non-critical edge of G_i. Let $G_{i+1} := G_i - e_i$.
4. Update $i := i + 1$ and go to step 2.

Note that at step 3, given some $i \geq 0$, it holds that $\alpha_\pi(G_i) = \alpha_\pi(G_{i+1})$, because a non-critical edge was deleted from G_i to obtain G_{i+1}. This implies $\alpha_\pi(G_0) = \alpha_\pi(G_0 - \bar{E})$ for all $\bar{E} \subseteq E_0 \setminus E(T)$.

We remark that only step 3 differs from [17,18]: In order to obtain a connected α_π-critical subgraph $T = (V_0, E(T))$ of G_0, it suffices to choose an arbitrarily non-critical edge of G_i. Due to this modification, we can prove that every edge $e \in E(T)$ is contained in some odd induced cycle H_e in G_0, i.e., $e \in E(H_e)$.

Assume there exists an edge $e \in E(T)$ that is *not* contained in an odd induced cycle in G_0. Let $\bar{E} \subseteq E(G_0) \setminus E(T)$ be a minimal subset of deleted edges such that e is critical in $G_0 - \bar{E}$. Note that such a set $\bar{E}$ exists since e is critical in T and $E(T) \subseteq E \setminus \bar{E}$. Then, $\bar{E} \cup \{e\}$ is critical since $\alpha_\pi(G_0 - \bar{E} - e) > \alpha_\pi(G_0 - \bar{E}) = \alpha_\pi(G_0)$. Furthermore, the critical set $\bar{E} \cup \{e\}$ is minimal since $\alpha_\pi(G_0 - (\bar{E} - e') - e) = \alpha_\pi(G_0 - (\bar{E} - e')) = \alpha_\pi(G_0)$ for all $e' \in \bar{E}$. Then $\bar{E} \cup \{e\}$ is a minimal critical set of edges in G_0 and by Lemma 2 there exists an edge $e_h \in \bar{E} \cup \{e\}$ that is part of an odd induced cycle in G_0. Before e_h was removed from G_0 both edges e and e_h were not critical and, hence, we should have deleted the edge e before edge e_h. This is a contradiction to the fact that e is an edge in T and, therefore, was not removed at all, whereas e_h was removed. □

With Lemma 3 we are finally ready to prove one of our main results.

Theorem 6. *Let $G = (V, E)$ be a graph, let $E' \subseteq E$, and define $G' := (V, E')$. If G' contains all odd induced cycles of G, then $\mathrm{DW}(G, G') = \mathrm{STAB}(G)$ holds.*

Proof. Let $\sum_{v\in V}\pi_v x_v \le \pi_0$ be some non-trivial, non-edge facet of STAB(G). Let $V_0 := \{v \in V : \pi_v > 0\}$ and let $G_0 := G[V_0] = (V_0, E_0)$ be the induced subgraph on the nodes with positive weights. Then G_0 with weights $\pi_v > 0$ for all $v \in V_0$ is a facet-graph and $|V_0| \ge 3$ holds.

Let $T_0 = (V_0, E(T_0))$ be a connected α_π-critical subgraph of G_0 such that every edge $e \in E(T_0)$ is contained in some odd induced cycle in G_0. Such a graph exists due to Lemma 3. Since G_0 is an induced subgraph of G, every induced cycle in G_0 is also induced in G. Hence, by hypothesis all edges of T_0 are contained in E' and T_0 is a subgraph of G'.

Since $\pi_0 = \alpha_\pi(G_0) = \alpha_\pi(T_0)$, the inequality $\sum_{v\in V_0}\pi_v x_v \le \pi_0$ is valid for STAB(T_0). Since T_0 is a subgraph of G', the inequality is also valid for STAB(G'), and therefore also for DW(G, G') $\subseteq$ STAB(G'). The facet $\sum_{v\in V}\pi_v x_v \le \pi_0$ was chosen arbitrarily, which implies that DW(G, G') $\subseteq$ STAB(G) and therefore DW(G, G') $=$ STAB(G) holds. □

This allows us to characterize the strongest possible Dantzig–Wolfe reformulations.

Theorem 7. *Let $G = (V, E)$ be a graph, let $E' \subseteq E$, and define $G' := (V, E')$. Then* DW(G, G') $=$ STAB(G) *if and only if G' contains all odd induced cycles of G.*

Proof. Follows directly from Theorems 3 and 6. □

5. A polyhedral result for the stable set polytope

When working on this paper it was pointed out to us by Bruce Reed that Theorem 7 can be stated as a previously unproven result for the stable set polytope.

A pair of non-adjacent nodes (v, w) is called an *odd pair* in G if all induced v-w-paths have odd length. Such pairs were particularly studied in the context of perfect graphs and the proof of the strong perfect graph theorem. Note that (v, w) is an odd pair in G if and only if there exists no odd induced cycle in $G+vw$ containing the edge vw. This leads us to

Theorem 8. *Let $G = (V, E)$ be a graph and let $v, w \in V$ with $vw \notin E$. It holds that (v, w) is an odd pair if and only if*

$$\mathrm{STAB}(G + vw) = \{x \in \mathrm{STAB}(G) : x_v + x_w \le 1\}. \tag{5}$$

Proof. By definition of the Dantzig–Wolfe polytope, the right hand side of Eq. (5) equals

$$\mathrm{DW}(G + vw, G) = \{x \in \mathrm{STAB}(G) : x_v + x_w \le 1\}.$$

By Theorem 7, the Dantzig–Wolfe polytope DW($G + vw, G$) is equal to STAB($G + vw$) if and only if G contains all odd induced cycles of $G + vw$. This is the case if and only if (v, w) is an odd pair in G, which proves the theorem. □

Theorem 8 was most recently proven using a different technique unrelated to Dantzig–Wolfe reformulations by Witt et al. [22]. This again implies Theorem 7. It is still important to keep our (alternative) proof of Theorem 7 for its insights into the structure of facet-graphs and the separation of cutting planes for the stable set problem.

6. Dantzig–Wolfe reformulations for related problems

In this section we investigate the question as to what extent similar proof ideas are useful for characterizing strongest and weakest Dantzig–Wolfe reformulations of other combinatorial optimization problems.

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6.1. The clique and the node covering problem

Theorem 7 immediately translates to the clique and the node covering problem because of their close proximity to the stable set problem.

A (fractional) clique in a graph $G = (V, E)$ is a (fractional) stable set in the complement graph $\overline{G} = (V, \overline{E})$ and vice versa. This is why $\text{CLIQUE}(G) = \text{STAB}(\overline{G})$ holds, where $\text{CLIQUE}(G)$ denotes the convex hull of incidence vectors of cliques in G. Analogously, this equation also holds for the corresponding fractional polytopes, i.e., $\text{FCLIQUE}(G) = \text{FRAC}(\overline{G})$, where

$$\text{FCLIQUE}(G) := \{x \in [0,1]^V : x_u + x_v \leq 1 \ \forall uv \in \overline{E}\}.$$

Hence, our results extend to the clique problem.

The complement of a (fractional) stable set is a (fractional) node cover and vice versa. Hence, $x \in \text{STAB}(G)$ if and only if $\mathbf{1} - x \in \text{NC}(G)$, where $\text{NC}(G)$ is the convex hull of incidence vectors of node covers in G and $\mathbf{1}$ is the vector of suitable dimension having all entries equal to one. The same holds for the fractional counterparts, the fractional stable set polytope $\text{FRAC}(G)$ and the fractional node covering polytope

$$\text{FNC}(G) := \{x \in [0,1]^V : x_u + x_v \geq 1 \ \forall uv \in E\}.$$

Since the function $f : [0,1]^V \to [0,1]^V, x \mapsto \mathbf{1} - x$ is an affine isomorphism, affine independence is preserved using this mapping. This implies that $\pi^T x \leq \pi_0$ is a facet of the (fractional) stable set polytope if and only if $\pi^T(\mathbf{1} - y) \leq \pi_0$ (or equivalently $\pi^T y \geq \pi^T \mathbf{1} - \pi_0$) is a facet of the (fractional) node covering problem. This is why our results extend to the node covering problem.

6.2. The matching problem

Once our argumentation is in place, we can see other polyhedral results from the literature through this lens. In particular, we exemplarily show how complete descriptions of integer hulls can be employed.

Let $G = (V, E)$ be a graph with $m := |E|$ edges. A set of edges $M \subseteq E$ is called *matching* in G if no edges of M are adjacent in G, i.e., $|\{e \in M : v \in e\}| \leq 1$ for all $v \in V$. Let $\mathcal{M}(G)$ be the set of all matchings in G. Given a weighting $w \in \mathbb{Z}^E_{\geq 0}$ on the edges, a matching M^* in G with weighting w is called *maximum weighted matching* if $M^* \in \text{argmax}\{\sum_{e \in M} w_e : M \in \mathcal{M}(G)\}$.

The maximum weighted matching problem can be formulated as an integer linear program by introducing a binary variable $x_e \in \{0, 1\}$ for each edge $e \in E$ indicating whether edge e is in the matching ($x_e = 1$) or not ($x_e = 0$). Furthermore, we have a constraint for each node $v \in V$ enforcing that at most one edge in the matching is incident to node v. The objective is to maximize the weight of the matching. This leads to the so-called *node formulation*:

$$\begin{aligned} \max \quad & \sum_{e \in E} w_e \cdot x_e \\ \text{s.t.} \quad & x(\delta(v)) \leq 1 \qquad \forall v \in V \\ & x \in \{0,1\}^E, \end{aligned}$$

where $\delta(v) := \{e \in E : v \in e\}$ for $v \in V$ denotes the set of incident edges to v and $x(\tilde{E}) := \sum_{e \in \tilde{E}} x_e$ for $\tilde{E} \subseteq E$. Note that $\delta(v)$ refers to the set of edges incident to v in a particular graph G. We will use the notation $\delta_G(v)$ whenever it is ambiguous which graph is referred to.

The *matching polytope* $\text{MP}(G)$ is defined as the convex hull of incidence vectors of matchings in G, i.e.,

$$\text{MP}(G) := \text{conv}\{x \in \{0,1\}^E : x(\delta(v)) \leq 1 \ \forall v \in V\}.$$

The set of LP-feasible solutions for the node formulation is denoted by $\mathrm{FMP}(G)$ and is defined as

$$\mathrm{FMP}(G) := \{x \in [0,1]^E : x(\delta(v)) \le 1 \ \forall v \in V\}.$$

We refer to $\mathrm{FMP}(G)$ as the *fractional matching polytope*.

Let $V' \subseteq V$ be a subset of the nodes of G. We define

$$\mathrm{MP}(G, V') := \mathrm{conv}\left\{x \in \{0,1\}^E : x(\delta(v)) \le 1 \ \forall v \in V'\right\}.$$

Note that the polytope $\mathrm{MP}(G, V')$ has the same dimension as the polytope $\mathrm{MP}(G)$, but only the degree constraints corresponding to nodes in V' are considered in the definition of the polytope $\mathrm{MP}(G, V')$. Obviously, it holds that $\mathrm{MP}(G, V) = \mathrm{MP}(G)$. Similarly, we define the corresponding fractional polytope

$$\mathrm{FMP}(G, V') := \left\{x \in [0,1]^E : x(\delta(v)) \le 1 \ \forall v \in V'\right\}.$$

Following Dantzig–Wolfe reformulation, we will convexify all constraints of the node formulation corresponding to nodes in V':

$$\mathrm{DW}(G, V') := \left\{x \in [0,1]^E : x \in \mathrm{MP}(G, V'),\ x(\delta(v)) \le 1 \ \forall v \in V \setminus V'\right\}.$$

Before we consider weakest and strongest possible Dantzig–Wolfe reformulations for the matching problem, we repeat known results on the matching problem and the corresponding polytopes from the literature.

The following theorem plays a role identical to Theorem 1 for the stable set problem.

Theorem 9 (*[23]*). *Let $G = (V, E)$ be a graph. Then,* $\mathrm{FMP}(G) = \mathrm{MP}(G) \iff$ *G is bipartite.*

In contrast to the stable set polytope, a complete description for the matching polytope is known.

Theorem 10 (*[24]*). *Let $G = (V, E)$ be a graph. The matching polytope* $\mathrm{MP}(G)$ *can be described using the node inequalities as well as the so-called* blossom inequalities*:*

$$\mathrm{MP}(G) = \Big\{x \in [0,1]^E : x\left(\delta_G(v)\right) \le 1 \ \forall v \in V,$$
$$x\left(E(S)\right) \le \frac{|S|-1}{2} \ \forall S \subseteq V,\ |S| \text{ odd}\Big\},$$

where $E(S) := E\left(G[S]\right)$ for $S \subseteq V$.

In fact, not all blossom inequalities are necessary for a complete description of the matching polytope. The following class of graphs will be important to see this. A graph $G = (V, E)$ is called *factor-critical* [25] if $G[V \setminus \{v\}]$ has a perfect matching for each $v \in V$. For example, every odd cycle is factor-critical. Note that each factor-critical graph has an odd number of nodes. With this definition at hand, we can state the following result proven by Edmonds [25]:

Theorem 11 (*[25]*). *Let $G = (V, E)$ be a graph and let $S \subseteq V$ with $|S|$ odd. Then $x(E(S)) \le \frac{|S|-1}{2}$ is a facet of* $\mathrm{MP}(G)$ *if and only if $G[S]$ is 2-connected and factor-critical.*

In contrast to the stable set problem, we used a modified polytope, namely $\mathrm{MP}(G, V')$, instead of the matching polytope of a subgraph to define the Dantzig–Wolfe polytope $\mathrm{DW}(G, V')$. The following lemma will be helpful when working with the polytope $\mathrm{MP}(G, V')$ and its fractional counterpart $\mathrm{FMP}(G, V')$.

Lemma 4. *Let $G = (V, E)$ and let $G' := G[V'] = (V', E')$ be the subgraph induced by $V' \subseteq V$. Then there exists a graph $\tilde{G} = (\tilde{V}, \tilde{E})$ with (i) $\tilde{V} \supseteq V'$, (ii) $\tilde{G}[V'] = G'$, and (iii) $d_{\tilde{G}}(v) := |\delta_{\tilde{G}}(v)| \le 1$ for $v \in \tilde{V} \setminus V'$, such that the polytopes* $\mathrm{FMP}(G, V')$ *and* $\mathrm{FMP}(\tilde{G})$ *as well as* $\mathrm{MP}(G, V')$ *and* $\mathrm{MP}(\tilde{G})$ *are isomorphic.*

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Proof. Let $\delta_G(V') := \bigcup_{v\in V'}\delta_G(v)$. Note that $E' \subseteq \delta_G(V')$. For each $e = uv \in \delta_G(V') \setminus E'$ with $v \in V \setminus V'$ define a new node v_e and a new edge $\tilde{e} := uv_e$. For each $e = uv \in E \setminus \delta_G(V')$ define new nodes u_e and v_e as well as a new edge $\tilde{\tilde{e}} := u_e v_e$. Let $\tilde{G} := (\tilde{V}, \tilde{E})$ be the graph with

$$\begin{aligned}\tilde{V} &:= V' \cup \{v_e : e = uv \in \delta_G(V') \setminus E', v \in V \setminus V'\}\\ &\quad \cup \{u_e, v_e : e = uv \in E \setminus \delta_G(V')\}\\ \tilde{E} &:= E' \cup \{\tilde{e} : e \in \delta_G(V') \setminus E'\} \cup \{\tilde{\tilde{e}} : e \in E \setminus \delta_G(V')\}.\end{aligned}$$

With this definition of $\tilde{G}$ it is easy to see that the polytopes $\mathrm{FMP}(G, V')$ and $\mathrm{FMP}(\tilde{G})$ as well as $\mathrm{MP}(G, V')$ and $\mathrm{MP}(\tilde{G})$ are isomorphic. □

Using Lemma 4, Theorem 9 can be extended to the polytopes $\mathrm{FMP}(G, V')$ and $\mathrm{MP}(G, V')$ as follows:

Lemma 5. *Let $G = (V, E)$ be a graph and let $G' := G[V'] = (V', E')$ be the subgraph of G induced by $V' \subseteq V$. Then, $\mathrm{FMP}(G, V') = \mathrm{MP}(G, V') \iff G'$ is bipartite.*

Furthermore, we can derive a complete description for $\mathrm{MP}(G, V')$ using Theorems 10 and 11 in combination with Lemma 4:

Corollary 1. *Let $G = (V, E)$ be a graph and let $V' \subseteq V$ be a subset of nodes. The polytope $\mathrm{MP}(G, V')$ can be described as follows:*

$$\begin{aligned}\mathrm{MP}(G, V') = \Big\{x \in [0,1]^E :\ & x(\delta_G(v)) \le 1\ \forall v \in V',\\ & x(E(S)) \le \frac{|S|-1}{2}\ \forall S \subseteq V',\\ & G[S]\text{2-connected and factor-critical}\Big\}.\end{aligned} \tag{6}$$

By means of the results for the polytopes $\mathrm{MP}(G, V')$ and $\mathrm{FMP}(G, V')$, we can investigate Dantzig–Wolfe reformulations for the matching problem: The following theorem makes an analogous statement to Theorem 2 and characterizes the weakest possible Dantzig–Wolfe reformulations of the node formulation for the matching problem:

Theorem 12. *Let $G = (V, E)$ be a graph and let $V' \subseteq V$ be a subset of nodes. Then, $\mathrm{FMP}(G) = \mathrm{DW}(G, V')$ holds if and only if G' is bipartite.*

Proof. Necessity follows from Geoffrion's result [2] in combination with Lemma 5.

For sufficiency suppose $\mathrm{FMP}(G) = \mathrm{DW}(G, V')$ holds and assume that the graph G' is not bipartite. Then there exists an odd cycle H in G'. Let $\bar{x}$ be the solution with

$$\bar{x}_e = \begin{cases}\frac{1}{2} & \text{if } e \in E(H)\\ 0 & \text{otherwise.}\end{cases}$$

The solution $\bar{x}$ is obviously in $\mathrm{FMP}(G)$ but not in $\mathrm{MP}(G, V')$, because it does not satisfy the blossom inequality corresponding to $V(H)$. Since $\mathrm{MP}(G, V') \supseteq \mathrm{DW}(G, V')$, this implies $\bar{x} \notin \mathrm{DW}(G, V')$. Therefore, $\mathrm{DW}(G, V') \ne \mathrm{FMP}(G)$, which is a contradiction to the assumption $\mathrm{DW}(G, V') = \mathrm{FMP}(G)$. □

Similar to Theorem 2 for the stable set problem, Theorem 12 (in combination with Lemma 5) shows that Geoffrion's necessary condition [2] for $P_{DW}(I') \subsetneq P_{LP}$ is sufficient when considering the node formulation for the matching problem.

On the other hand, strongest possible Dantzig–Wolfe reformulations of the node formulation for the matching problem are characterized as follows:

Theorem 13. *Let $G = (V, E)$ be a graph and let $V' \subseteq V$ be a subset of nodes. Then* $\mathrm{MP}(G) = \mathrm{DW}(G, V')$ *holds if and only if G' contains all 2-connected, factor-critical induced subgraphs of G.*

Proof. If G' contains all 2-connected, factor-critical induced subgraphs of G, then Corollary 1 implies that $\mathrm{MP}(G) = \mathrm{DW}(G, V')$.

Suppose $\mathrm{MP}(G) = \mathrm{DW}(G, V')$ holds and assume there exists a 2-connected, factor-critical induced subgraph $G[S]$ of G for some $S \subseteq V$ with $S \nsubseteq V'$. Hence, there exists some $v \in S$ with $v \notin V'$. Since $G[S]$ is factor-critical, there exists some vertex $x \in \mathrm{FMP}(G[S])$ with $x \in \{0, \frac{1}{2}, 1\}^{E(S)}$ and $x(E(S)) = \frac{|S|}{2}$ for which $\{e \in E(S) : x_e = \frac{1}{2}\}$ consists of a single cycle containing node v [26,27]. Let $\{e_i = v_i v_{i+1} : i = 1, \dots, 2k\} \cup \{e_{2k+1} = v_1 v_{2k+1}\} \subseteq E(S)$ with $v = v_1$ be this cycle for some $k \in \mathbb{Z}_{>0}$. The solution $\bar{x} \in [0, 1]^E$ with $\bar{x}_e = x_e$ for $e \in E(S)$ and $\bar{x}_e = 0$ otherwise is obviously not in $\mathrm{MP}(G)$, because the blossom inequality corresponding to S is not satisfied. The solution $\bar{x}$ is a convex combination of incidence vectors x^{even} and x^{odd} of the edge sets $M_{even} := \{e_{2\ell} : \ell, \dots, k\} \cup \{e \in E : \bar{x}_e = 1\}$ and $M_{odd} := \{e_{2\ell+1} : \ell, \dots, k\} \cup \{e \in E : \bar{x}_e = 1\}$ using coefficients $\frac{1}{2}$ for both incidence vectors, i.e., $\bar{x} = \frac{1}{2} x^{even} + \frac{1}{2} x^{odd}$. Notice that M_{even} is a matching in G, and hence, $x^{even} \in \mathrm{MP}(G) \subseteq \mathrm{DW}(G, V')$. On the contrary, M_{odd} is no matching in G since $e_1 = v_1 v_2 \in M_{odd}$ and $e_{2k+1} = v_{2k+1} v_1 \in M_{odd}$. Nevertheless, it holds that $x^{even} \in \mathrm{DW}(G, V')$ because there is no degree constraint corresponding to node $v = v_1$ in a full description of $\mathrm{MP}(G, V')$ (compare Corollary 1). Hence, $\bar{x} \in \mathrm{DW}(G, V')$, which contradicts the assumption $\mathrm{MP}(G) = \mathrm{DW}(G, V')$. □

We have seen that we can characterize weakest and strongest possible Dantzig–Wolfe reformulations for the matching problem using results from the literature. In particular, we exploited that a complete description of the matching polytope is known in order to prove the sufficient condition for strongest possible Dantzig–Wolfe reformulations. Besides, the structure of most proofs is similar to the structure of the corresponding proofs for the stable set problem. In the same way, one may derive similar results for other problems for which a complete description of the corresponding integer hull is known.

6.3. The set packing problem

The set packing problem is intimately related to the stable set problem. Nevertheless, characterizing weakest and strongest possible Dantzig–Wolfe reformulations for the set packing problem seems to be more challenging, as we will see next.

Let $A \in \{0, 1\}^{m \times n}$ and $c \in \mathbb{Q}^n$. The set packing problem is the following binary program:

$$\begin{array}{lll} (SP) & \max & c^T x \\ & \text{s. t.} & Ax \leq \mathbf{1} \\ & & x \in \{0, 1\}^n. \end{array}$$

The *set packing polytope* $\mathrm{SP}(A)$ is defined as

$$\mathrm{SP}(A) := \mathrm{conv}\{x \in \{0, 1\}^n : Ax \leq \mathbf{1}\}.$$

The set of LP-feasible solutions to (SP) is denoted by $\mathrm{FSP}(A)$ and is defined as

$$\mathrm{FSP}(A) := \{x \in [0, 1]^n : Ax \leq \mathbf{1}\}.$$

The polytope $\mathrm{FSP}(A)$ is called *fractional set packing polytope*. Furthermore, we call solutions $\bar{x} \in \mathrm{SP}(A)$ *set packings* and solutions $\bar{x} \in \mathrm{FSP}(A)$ *fractional set packings*. Let $I' \subseteq I := \{1, \dots, m\}$ and let $A' := A_{I'} \in \{0, 1\}^{|I'| \times n}$ be the matrix that consists of the rows with index in I'. Analogously to the stable set problem, we define the Dantzig–Wolfe polytope $\mathrm{DW}(A, A')$ corresponding to the Dantzig–Wolfe reformulation of (SP), where the constraints $A'x \leq \mathbf{1}$ are convexified, as

$$\mathrm{DW}(A, A') := \left\{x \in [0,1]^n : A_{I\setminus I'}x \leq \mathbf{1},\, x \in \mathrm{SP}(A')\right\}.$$

A matrix $A \in \{0,1\}^{m\times n}$ is called *perfect* [28,29] if and only $\mathrm{FSP}(A) = \mathrm{SP}(A)$.

We denote with A_{ij} the entry of matrix A in the ith row and jth column; with $A_{i\cdot}$ we denote the row vector corresponding to the ith row. A row $A_{r\cdot}$ of A is *dominated* [30] by row $A_{s\cdot}$ of A with $r \neq s$ if $A_{r\cdot} \leq A_{s\cdot}$ holds, where the comparison is componentwise. Let $G(A) = (V(A), E(A))$ be the *conflict graph* [30] of A, i.e., $V(A) = \{1,\dots,n\}$ and $uv \in E(A)$ if and only if there exists a row r such that $A_{ru} = A_{rv} = 1$. Note that $\mathrm{SP}(A) = \mathrm{STAB}(G(A))$ and $\mathrm{FSP}(A) \subseteq \mathrm{FRAC}(G(A))$ [15]. This is why we sometimes refer to set packings as stable sets in the conflict graph. Using these observations, we can derive the following subset relation for the Dantzig–Wolfe polytopes of the respective problems:

$$\begin{aligned}
\mathrm{DW}(A, A') &= \left\{x \in [0,1]^n : A_{I\setminus I'}x \leq \mathbf{1},\, x \in \mathrm{SP}(A')\right\}\\
&= \left\{x \in [0,1]^n : x \in \mathrm{FSP}(A_{I\setminus I'}),\, x \in \mathrm{SP}(A')\right\}\\
&= \left\{x \in [0,1]^n : x \in \mathrm{FSP}(A_{I\setminus I'}),\, x \in \mathrm{STAB}(G(A'))\right\}\\
&\subseteq \left\{x \in [0,1]^n : x \in \mathrm{FRAC}(G(A_{I\setminus I'})),\, x \in \mathrm{STAB}(G(A'))\right\}\\
&= \{x \in [0,1]^n : x_u + x_v \leq 1\ \forall uv \in E(A_{I\setminus I'}),\, x \in \mathrm{STAB}(G(A'))\}\\
&= \mathrm{DW}(G(A), G(A')).
\end{aligned}$$

A matrix $A \in \{0,1\}^{m\times n}$ is a *clique-node incidence matrix* of graph $G = (V, E)$ with $n = |V|$ if there is a one-to-one correspondence between the maximal cliques of G and the rows of A such that for each maximal clique $Q \subseteq G$ there exists some row j of A with $A_{ij} = 1$ for $i \in V(Q)$ and $A_{ij} = 0$ for $i \in V \setminus V(Q)$.

Perfect matrices are closely related to perfect graphs:

Theorem 14 (*[31]*)**.** *A matrix $A \in \{0,1\}^{m\times n}$ is perfect if and only if its non-dominated rows form the clique-node incidence matrix of a perfect graph.*

In the following, we will investigate Dantzig–Wolfe reformulations for the set packing problem using the results for the stable set problem and similar proof ideas.

6.3.1. Weakest possible Dantzig–Wolfe reformulations

When investigating weakest possible Dantzig–Wolfe reformulations for the set packing problem, the results obtained for the stable set problem are unfortunately not helpful, although the corresponding fractional polytopes are related to each other. We remark that the fractional set packing polytope $\mathrm{FSP}(A)$ can be obtained by adding "some" clique inequalities to the description of the fractional stable set polytope $\mathrm{FRAC}(G(A))$.

Nevertheless, we can use Geoffrion's result in combination with the definition of perfect matrices in order to obtain the following sufficient condition for $\mathrm{DW}(A, A') = \mathrm{FSP}(A)$:

Corollary 2. *Let $A \in \{0,1\}^{m\times n}$ be a matrix and let A' be a submatrix of A with $A' = A_{I'}$ for some $I' \subseteq I := \{1,\dots,m\}$. If A' is perfect, then $\mathrm{DW}(A, A') = \mathrm{FSP}(A)$ holds.*

Proof. Follows from Geoffrion [2] and the definition of perfect graphs. □

In contrast to the specialization of Geoffrion's necessary condition for $P_{DW}(I) \subsetneq P_{LP}$ [2] in context of the stable set problem, this is not a sufficient condition for $P_{DW}(I) \subsetneq P_{LP}$ in the context of the set packing problem. Note that constraints in $A_{I\setminus I'}x \leq \mathbf{1}$ can be much stronger than the edge constraints $x_u + x_v \leq 1$ for $uv \in E \setminus E'$.

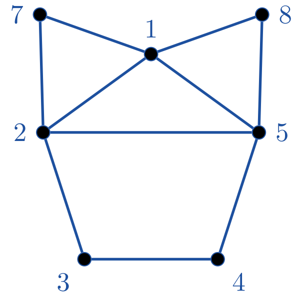

Fig. 2. Example of a conflict graph $G(A)$ belonging to a set packing problem, in which the condition of Proposition 1 is satisfied, but the condition of Corollary 2 is not.

Consider the example of Fig. 2: Suppose the rows of A correspond to the maximal cliques in the graph depicted in Fig. 2. When A' consists of all constraints but the constraint corresponding to the clique $G[\{1,2,5\}]$, then $\mathrm{DW}(A,A') = \mathrm{SP}(A)$. Note that $G(A')$ is not perfect because of the odd induced cycle consisting of the node set $\{1,2,3,4,5\}$. This example shows that the condition of Corollary 2 is not a necessary condition for $\mathrm{DW}(A,A') = \mathrm{FSP}(A)$.

Nevertheless, we can generalize the sufficient condition for $P_{DW}(I) = P_{LP}$ used in Corollary 2 as follows:

Proposition 1. *Let $A \in \{0,1\}^{m\times n}$ be a matrix and let A' be a submatrix of A with $A' = A_{I'}$ for some $I' \subseteq I := \{1,\dots,m\}$. If there exists a perfect graph $\tilde{G}$ with $G(A') \subseteq \tilde{G} \subseteq G(A)$ and all clique inequalities corresponding to cliques in $\tilde{G}$ are dominated by constraints in $Ax \le \mathbf{1}$, then $\mathrm{DW}(A,A') = \mathrm{FSP}(A)$ holds.*

Proof. Let $\tilde{A}$ be the clique-node incidence matrix of $\tilde{G}$. Because $G(A')$ is a subgraph of $\tilde{G}$, it holds $\mathrm{STAB}(G(A')) \supseteq \mathrm{STAB}(\tilde{G})$ and

$$\mathrm{SP}(A') \supseteq \mathrm{SP}(\tilde{A}). \tag{7}$$

Furthermore, it holds that

$$\mathrm{SP}(\tilde{A}) = \mathrm{FSP}(\tilde{A}) \tag{8}$$

since $\tilde{A}$ is the clique-node incidence matrix of the perfect graph $\tilde{G}$.

Since all clique inequalities corresponding to cliques in $\tilde{G}$ are dominated by constraints in $Ax \le \mathbf{1}$, the following holds

$$\{x \in [0,1]^n : Ax \le \mathbf{1}\} = \{x \in [0,1]^n : Ax \le \mathbf{1},\ \tilde{A}x \le \mathbf{1}\}. \tag{9}$$

Finally, the following subset relations complete the proof:

$$\begin{aligned}
\mathrm{FSP}(A) &= \{x \in [0,1]^n : Ax \le \mathbf{1}\}\\
&\overset{(9)}{=} \{x \in [0,1]^n : Ax \le \mathbf{1},\ \tilde{A}x \le \mathbf{1}\}\\
&= \{x \in [0,1]^n : Ax \le \mathbf{1},\ x \in \mathrm{FSP}(\tilde{A})\}\\
&\overset{(8)}{=} \{x \in [0,1]^n : Ax \le \mathbf{1},\ x \in \mathrm{SP}(\tilde{A})\}\\
&\overset{(7)}{\subseteq} \{x \in [0,1]^n : A_{I\setminus I'}x \le \mathbf{1},\ x \in \mathrm{SP}(A')\}\\
&= \mathrm{DW}(A,A')\\
&\subseteq \mathrm{FSP}(A). \quad \square
\end{aligned}$$

Notice that in an arbitrary graph $\tilde{G} \subseteq G(A)$ there can exist clique inequalities corresponding to cliques in $\tilde{G}$ that are not dominated by constraints in $Ax \leq \mathbf{1}$ only if A is no clique-node incidence matrix. Furthermore, we remark that the set packing problem from Fig. 2 satisfies the condition of Proposition 1 (by choosing $\tilde{G} = G(A)$, which is a perfect graph).

The sufficient condition for $\mathrm{DW}(A, A') = \mathrm{FSP}(A)$ used in Proposition 1 is related to the *perfect graph sandwich problem*. Given the edge set E of a graph G, let $\overline{E}$ denote the edge set of the complement graph $\overline{G}$. The *sandwich problem for property Π* [32,33] is the following problem: Given a graph $G = (V, E)$ and a set $E_0 \subseteq \overline{E}$ of (optional) additional edges, we ask whether there exists a graph $G' = (V, E')$ with $E \subseteq E' \subseteq E \cup E_0$ satisfying the desired property Π (such as being perfect in our case). The complexity of the perfect graph sandwich problem is still open [32,33]. This is one of the reasons why this problem seems difficult to tackle. Unfortunately, we were not yet able to find a counterexample or prove that the sufficient condition for $\mathrm{DW}(A, A') = \mathrm{FSP}(A)$ used in Proposition 1 is also a necessary condition.

6.3.2. Strongest possible Dantzig–Wolfe reformulations

The characterization of strongest possible Dantzig–Wolfe reformulations for the stable set problem can be helpful when investigating strongest possible Dantzig–Wolfe reformulations for the set packing problem because the corresponding integer hulls are identical. Using the observations on the correspondence between the polytopes introduced in the context of the stable set and the set packing problem, we can easily derive the following sufficient condition for $\mathrm{DW}(A, A') = \mathrm{SP}(A)$:

Corollary 3. *Let $A \in \{0,1\}^{m \times n}$ be a matrix and let A' be a submatrix of A with $A' = A_{I'}$ for some $I' \subseteq I := \{1, \ldots, m\}$. If for every $e \in E(A)$ that is contained in an odd induced cycle of $G(A)$ it holds that $e \in E(A')$, then $\mathrm{DW}(A, A') = \mathrm{SP}(A)$ holds.*

Proof. Under the hypothesis, Theorem 7 implies

$$\mathrm{DW}(G(A), G(A')) = \mathrm{STAB}(G(A)). \tag{10}$$

The following subset relations complete the proof:

$$\begin{aligned} \mathrm{SP}(A) &\subseteq \mathrm{DW}(A, A') \\ &\subseteq \mathrm{DW}(G(A), G(A')) \\ &\overset{(10)}{=} \mathrm{STAB}(G(A)) \\ &= \mathrm{SP}(A). \quad \square \end{aligned}$$

An *antihole* is the complement graph of an odd hole. Let $H = (V_H, E_H)$ be an odd antihole in $G(A)$ with $|V_H| = 2k + 1$ for some $k \in \mathbb{Z}$ with $k \geq 2$. The following *odd antihole inequality* [15] is valid for the set packing polytope $\mathrm{SP}(A)$ (and for the stable set polytope $\mathrm{STAB}(G(A))$):

$$\sum_{v \in V_H} x_v \leq 2.$$

Using this together with a similar proof idea as in the context of the stable set problem, we can derive the following necessary condition for $\mathrm{DW}(A, A') = \mathrm{SP}(A)$:

Corollary 4. *Let $A \in \{0,1\}^{m \times n}$ be a matrix and let A' be a submatrix of A with $A' = A_{I'}$ for some $I' \subseteq I := \{1, \ldots, m\}$. If $\mathrm{DW}(A, A') = \mathrm{SP}(A)$ holds, then for every $e \in E(A)$ that is contained in an odd hole or odd antihole of $G(A)$ it holds that $e \in E(A')$.*

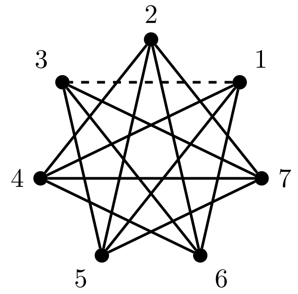

Fig. 3. Odd antihole $H = (V_H, E_H)$ for $k = 3$ used in the proof of Corollary 4. The solid edges are contained in $G(A')$, whereas the dashed edge is not.

Proof. Suppose that $\mathrm{DW}(A, A') = \mathrm{SP}(A)$ holds and assume there exists an odd hole $H = (V_H, E_H)$ of $G(A)$ that is not contained in $G(A')$, i.e., $E_H \not\subseteq E(A')$. Analogously to the proof of Theorem 3, we can create a solution $\bar{x} \in \mathrm{DW}(A, A')$ that does not satisfy the odd cycle inequality corresponding to the odd hole H such that $\bar{x} \notin \mathrm{SP}(A)$ holds. This contradicts the assumption $\mathrm{DW}(A, A') = \mathrm{SP}(A)$.

Now suppose that $\mathrm{DW}(A, A') = \mathrm{SP}(A)$ holds and assume there exists an odd antihole $H = (V_H, E_H)$ of $G(A)$ that is not contained in $G(A')$, i.e., $E_H \not\subseteq E(A')$. Hence, there exists an edge $e \in E_H$ with $e \notin E(A')$. Let $V_H = \{v_1, v_2, \ldots, v_{2k+1}\}$ and

$$E_H = E(A) \setminus (\{v_i v_{i+1} : i = 1, \ldots, 2k\} \cup \{v_1 v_{2k+1}\})$$

for some $k \in \mathbb{Z}_{>0}$. W.l.o.g., let $e = v_1 v_3$. The antihole is depicted in Fig. 3.

The solution $\bar{x}$ with

$$\bar{x}_v := \begin{cases} \dfrac{1}{2} & \text{if } v \in \{v_1, v_2, v_3, v_{2k}, v_{2k-1}\} \\ 0 & \text{otherwise,} \end{cases}$$

is obviously not in $\mathrm{SP}(A)$, because the odd antihole inequality $\sum_{v \in V_H} x_v \leq 2$ is not satisfied. The solution $\bar{x}$ is a convex combination of set packings x^1 and x^2 in $G(A')$ with

$$x^1_v := \begin{cases} 1 & \text{if } v = v_1, v_2, v_3 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$x^2_v := \begin{cases} 1 & \text{if } v = v_{2k}, v_{2k-1} \\ 0 & \text{otherwise,} \end{cases}$$

using coefficients $\frac{1}{2}$ for both set packings, i.e., $\bar{x} = \frac{1}{2}x^1 + \frac{1}{2}x^2$. Thus, $\bar{x} \in \mathrm{SP}(A')$ holds. Furthermore, the solution $\bar{x}$ satisfies $A\bar{x} \leq \mathbf{1}$ by construction. This implies that $\bar{x} \in \mathrm{DW}(A, A')$ holds, which contradicts the assumption that $\mathrm{DW}(A, A') = \mathrm{SP}(A)$. $\square$

Consider the conflict graph in Fig. 4 and suppose that we are given a matrix A that has a row for each 3-clique in the conflict graph.

In order to obtain a strongest possible Dantzig–Wolfe reformulation, it is not clear what to do with the (rows that induce) edges in the conflict graph that are contained in 3-cliques, but not in an odd hole or odd antihole. The set packing polytope $\mathrm{SP}(A)$ (or equivalently, the stable set polytope $\mathrm{STAB}(G(A))$) is completely described by the odd wheel inequality $\sum_{v=1}^{5} x_v + 2x_6 \leq 2$ and the 3-cliques. In this example, the reformulation is strongest possible, i.e., $\mathrm{DW}(A, A') = \mathrm{SP}(A)$, if and only if the odd wheel $G[\{1, \ldots, 6\}]$ is contained in the conflict graph $G(A')$.

Considering the following three cases, we should get an idea why it seems difficult to decide whether an edge that belongs to a 3-clique in the conflict graph $G(A)$ should be contained in the conflict graph $G(A')$:

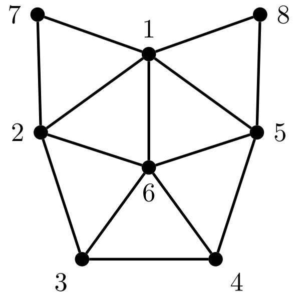

Fig. 4. Example of a conflict graph $G(A)$ belonging to a set packing problem, in which it is difficult to decide which conflicts that correspond to edges in 3-cliques in $G(A)$ should be represented in the conflict graph $G(A')$ in order to construct a strongest possible Dantzig–Wolfe reformulation.

- Choose A' such that $E(A') = E(A) \setminus \{\{1,7\},\{2,7\},\{1,8\},\{5,8\}\}$. In this case $\mathrm{DW}(A,A') = \mathrm{SP}(A)$ holds because the odd wheel is contained in the conflict graph $G(A')$. All odd holes but not all 3-cliques (the outer ones are missing) are represented in the conflict graph $G(A')$.
- Choose A' such that $E(A') = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{1,5\}\}$. In this case $\mathrm{DW}(A,A') \neq \mathrm{SP}(A)$ holds because the odd wheel is not contained in the conflict graph $G(A')$. All odd holes but not all 3-cliques (all 3-cliques are missing) are represented in the conflict graph $G(A')$.
- Choose A' such that $E(A') = E(A) \setminus \{\{1,6\}\}$. In this case $\mathrm{DW}(A,A') \neq \mathrm{SP}(A)$ holds because the odd wheel is not contained in the conflict graph $G(A')$. All odd holes but not all 3-cliques (some inner ones are missing) are represented in the conflict graph $G(A')$.

This example shows that we have to further differentiate the conflicts contained in 3-cliques of the conflict graph. In order to choose A' in such a way that $\mathrm{DW}(A,A') = \mathrm{SP}(A)$ holds, it is crucial whether the 3-clique $\{1,2,6\}$ or the 3-clique $\{1,2,7\}$ (analogously, $\{1,5,6\}$ or $\{1,5,8\}$) is represented in the conflict graph $G(A')$.

7. Discussion

Dantzig–Wolfe reformulation of an integer program naturally lends itself to a multitude of relaxations since we are free to choose the decomposition to apply to a model. In particular, we always have the (uninteresting) options to reformulate *all* constraints, thereby obtaining the integer hull, or to reformulate *no* constraint, leading to the LP relaxation. Consequently, asking for "the" strength of Dantzig–Wolfe reformulation is non-obvious in general. As a seminal step towards an answer, we focus on the textbook model of the stable set problem. It has been the object of study in previous investigations of the strength of several other relaxations.

In this paper, we characterize the two extreme cases. We have seen that we obtain a weakest possible Dantzig–Wolfe reformulation if and only if the convexified subgraph is bipartite. A Dantzig–Wolfe reformulation that is strongest possible is obtained if and only if the convexified subgraph contains all odd induced cycles of the original graph.

The proofs leading to Theorem 7 give new insights into the structure of facet-graphs and on the separation of cutting planes for the stable set problem. Sewell asked in his Ph.D. thesis [18] if every pair of critical edges in a facet-graph is contained in an odd induced cycle. We reconsidered this question and proved that every critical edge in a facet-graph is contained in an odd induced cycle. In addition, we generalized this result to critical sets of edges. Furthermore, we proved that every facet-graph contains a connected α_π-critical subgraph with the same set of nodes such that every edge of the connected α_π-critical subgraph is contained in an odd induced cycle of the initial facet-graph. When searching for facet-inducing cutting planes, we only have to consider the subgraph spanned by all edges contained in some odd induced cycle of the graph.

We see it as a contribution of independent interest that our results motivated viewing at polyhedral results from a different angle. The main theorem of Witt et al. [22], a polyhedral characterization of odd pairs in graphs, can be proven using our characterization of a strongest possible Dantzig–Wolfe decomposition (see proof of Theorem 8) and vice versa (see [22]).

Computational implications of our work may be interesting, but are out of the scope of this paper. If we wanted to apply the strongest possible Dantzig–Wolfe reformulation for the stable set problem in practice, we had to decide which edges are contained in an odd induced cycle. Unfortunately, deciding this for a given edge is an NP-complete problem already [34]. Moreover, the parameterized problem of deciding whether a given edge is contained in an odd hole of size at most k for some given parameter $k \in \mathbb{Z}_{\geq 3}$ is W[1]-complete [35]. Even if we managed to practically produce the reformulation, we expect the LP relaxation of the Dantzig–Wolfe reformulation to be too hard to solve effectively in practice. Then again, the actual strengthening of the formulation by Dantzig–Wolfe reformulation also depends on the objective function, and a theoretically weak reformulation may be strong in practice (and vice versa). From a theoretical point of view, a logical next step is to investigate Dantzig–Wolfe reformulations between the two extremes. An interesting family of reformulations for the stable set problem is obtained by convexifying all odd induced cycles of size at most $k \in \mathbb{Z}_{\geq 0}$, which may be fixed or dependent on the size of the graph. This may yield a provably strong formulation, while solving the LP relaxation of the Dantzig–Wolfe reformulation stays computationally tractable.

The most interesting question is how our work can be further developed or generalized. While we were able to extend our characterizations to intimately related problems, the still related but richer set packing problem defied our efforts (we were not able to close the gap between our necessary and sufficient conditions for weakest and strongest possible Dantzig–Wolfe reformulations). This will not become easier when models become more complex. For other combinatorial optimization problems or integer programs in general it is not clear whether a proper subset of constraints exists that, when convexified, yield the respective integer hull. So one would need to settle with less. Therefore, it would be interesting to investigate the strength of Dantzig–Wolfe reformulations relative to other relaxations, when both are applied to the same problem and model. Such an endeavor could theoretically support practical evidence that Dantzig–Wolfe reformulations cannot (much) be further strengthened by generic classes of cutting planes [36].

We hope that our work contributes to an understanding of and spawns further interest in “the” strength of Dantzig–Wolfe reformulations in general.

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