MaxFlow-MinCut Duality for a Paint Shop Problem

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Abstract. Motivated by an application in car manufacturing we consider the following problem: How can we synthesize a given word from restricted reservoirs of colored letters with a minimal number of color changes between adjacent letters?

We focus on instances in which each letter occurs exactly twice, once in each of two given colors. In this case the problem turns out to be the dual of a MinCut problem for one point extensions of a certain class of regular matroids.

We discuss consequences of the MaxFlow-MinCut duality and describe algorithmic approaches.

1 Introduction

A sequence of car bodies has to pass various shops during the production process, including a press and a body shop, a paint shop, and an assembly shop. The daily sequencing of the car bodies has to be done with respect to the minimization of the specific objective function of each of these shops. We focus on the paint shop, where the objective function consists in the minimization of the number of color changes that occur whenever two consecutive car bodies have to be colored in different colors, giving rise to non-negligible costs and water pollution.

The number of color changes may be reduced by the use of interim storage systems (see [1]) that permute a car body sequence before it enters the paint shop. However, the efficiency of succeeding production shops may also significantly decrease for the permuted sequence. We therefore consider the car body sequence to be a fixed external parameter.

Together with the fact that current technology heads for the detachment of car bodies and their features (what allows us to uncouple car bodies and enamel colors), the minimization of color changes for a car body sequence yields a new type of combinatorial problem.

We first give a formal problem description, a review of complexity results, and previous results and conjectures on the minimal number of color changes for regularly structured instances. Among these, we examine particular instances from a matroid point of view in more detail. Our notation is fairly standard.

2 Problem Formulation and Previous Results

We are given a fixed sequence of car bodies together with a set of orders that specify the demand of each car body type in each color. We assume that these orders are given by an inital coloring of the car body sequence. We associate a letter of an alphabet Σ with each car body type and represent a sequence of n car bodies by a word $w \in \Sigma^n$. A coloring of w is represented by a vector $f \in F^n$ for some color set F, where f_i denotes the color of w_i for all i. We say that we have a color change within f whenever $f_i \neq f_{i+1}$.

Our problem consists in finding a permutation that minimizes the number of color changes within f and leaves the sequence of letters in w unchanged.

Problem 1. Paint Shop Problem for Words (PPW)

Instance A finite alphabet Σ, a word $w = (w_1, \ldots, w_n) \in \Sigma^n$, a finite color set F, and a coloring $f = (f_1, \ldots, f_n) \in F^n$ of w.

Question Find a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $w_{\sigma(i)} = w_i$ for $i = 1, \ldots, n$, and the number of color changes within $\sigma(f) = (f_{\sigma(1)}, \ldots, f_{\sigma(n)})$ is minimized.

The following complexity results hold.

Theorem 1 ([2]). *Any instance of PPW can be solved by a dynamic program with a space and time complexity of $\mathcal{O}(|F|n^{(|F|-1)|\Sigma|})$. PPW is NP-complete for either $|F| = 2$ or $|\Sigma| = 2$, that is, if at least one parameter is unbounded.*

2.1 Upper Bounds for Regular Instances

We now turn to structured instances of PPW. In the following, we denote an instance of PPW by $(w; f)$, and the minimal number of color changes for $(w; f)$ by $\gamma(w; f)$. Recall that the coloring f determines reservoirs of colored letters for $(w; f)$. We denote the reservoir of letter i in color j by $R(i, j)$.

Definition 1. Given a fixed integer $k \geq 1$, we call an instance $(w; f)$ of PPW k-regular, if $R(i, j) = k = \frac{n}{|\Sigma||F|}$ for all letters i and colors j.

The following upper bounds on $\gamma(w; f)$ hold for k-regular instances.

Lemma 1 ([2]). *If $|\Sigma| = |F| = 2$, then $\gamma(w; f) \leq 2$. If $|\Sigma| = 2$, then $\gamma(w; f) \leq 2(|F| - 1)$.*

We conjecture the following general upper bound on $\gamma(w; f)$. A linear algebra argument yields the correctness of Conjecture 1 for $k = 1$.

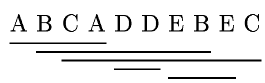
Conjecture 1 ([2]). For a k-regular instance holds $\gamma(w; f) \leq |\Sigma|(|F| - 1)$.

Theorem 2. *For a 1-regular instance holds $\gamma(w; f) \leq |\Sigma|(|F| - 1)$.*

Examples (see [2]) show that the bound given in Conjecture 1 is tight if the conjecture is correct. Natural solution approaches like a greedy coloring algorithm or an improvement algorithm based on color exchanges of letters or letter blocks fail to yield or even approximate an optimal solution in general.

3 1-Regular Instances and Two Colors

This section focuses on 1-regular instances with a color set of size $|F| = 2$, thus every letter is available exactly once in each color. We do not know whether PPW is polynomially solvable when restricted to such instances, but we give some indications that might support such a conjecture. In the following we denote by $\mathbf{0}$ resp. $\mathbf{1}$ the vector (of appropriate dimension) of all zeros resp. ones.



(a)

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

(b)

Fig. 1. An example instance and the associated matrix A

First we associate an interval $I(b)$ to each letter $b \in \Sigma$, where $I(b)$ runs from the first occurence of b in w to the second (see Figure 1(a)). We may consider each $I(b)$ being an interval on the real line. The restriction of PPW to 1-regular instances with $|F| = 2$ is then equivalent to the following problem.

Problem 2. Odd Intersection of Intervals

Instance A set of closed intervals on the real line, where no two intervals share a common endpoint.

Question Find a minimal set of points on the real line that intersects each interval in an odd number of points.

We denote the set of intervals of an instance $(w; f)$ of Problem 2 by $I(w; f)$ and interpret each point that intersects one or more intervals of $I(w; f)$ as a color change or a cut between two adjacent letters of w. In the most easiest case, $I(w; f)$ is a clutter.

Lemma 2. *If $I(w; f)$ is a clutter, then $\gamma(w; f)$ can be computed by a greedy algorithm.*

Otherwise, we consider the $(|\Sigma| \times n - 1)$-matrix A (siehe Figure 1(b)) that is defined by

$$a_{ij} := \begin{cases} 1 \text{ , if a cut between } w_j \text{ and } w_{j+1} \text{ intersects } I(i)\text{, and} \\ 0 \text{ , otherwise.} \end{cases}$$

Obviously A has the consecutive ones (CO) property for rows (see [3]). Thus, A is totally unimodular. Furthermore, A contains at least all maximal cliques of the interval graph of $I(w; f)$. Note that any instance of Problem 2 can be solved by the integer program depicted in Figure 2(a).

$$
\begin{aligned}
\mathbf{1}^T\boldsymbol{x} &\to \min!\\
A\boldsymbol{x} - 2I\boldsymbol{y} &= \mathbf{1}\\
x_i &\in \{0,1\}\\
y_i &\in N_{\geq 0}
\end{aligned}
\qquad\qquad
\begin{aligned}
\mathbf{1}^T\boldsymbol{x} &\to \min!\\
A\boldsymbol{x} - 2I\boldsymbol{y} &= \boldsymbol{b}\\
x_i &\in \{0,1\}\\
y_i &\in N_{\geq 0}
\end{aligned}
$$

(a) (b)

Fig. 2. Integer programs for the solution of Problem 2

3.1 Lower Bounds for the Number of Interval Cuts

The fact that A is a node-clique matrix of an interval graph enables us to compute a lower bound on the number of intersections of each interval in an optimal solution for Problem 2. Therefore we consider the partial ordering on $I(w;f)$ given by proper containment, i. e. $I(b') < I(b) :\Leftrightarrow I(b') \neq I(b)$ and $I(b')$ is properly contained in $I(b)$. We define the set $C(b) := \{I(b') : I(b') < I(b)\}$ for all $b \in \Sigma$ and assign a lower bound of 1 to all $I(b)$ with $C(b) = \emptyset$. Then, we compute a lower bound for an interval $I(b)$ whenever all intervals in $C(b)$ are already assigned a lower bound. Therefore we have to compute the value of a maximum weighted stable set in $C(b)$, where the weight of each interval is given by its lower bound. The result has to be rounded up to the next odd integer. Note that the maximum weighted stable set problem can be solved in polynomial time on interval graphs by computing a maximum weighted clique on its complement graph (see [3]).

Theorem 3. *Let P resp. P' denote the IP shown in Figure 2(a) resp. (b), and let $F(P)$ denote the set of vectors feasible for P. Then the set of vectors feasible for P' is given by $F(P') = \{(\boldsymbol{x}, \boldsymbol{y} - \frac{1}{2}(\boldsymbol{b} - \mathbf{1})) : (\boldsymbol{x}, \boldsymbol{y}) \in F(P)\}$.*

Thus a feasible solution for P' can be derived from a feasible solution for P (and vice versa) by changing only the vector $\boldsymbol{y}$. In the following we denote the vector of lower bounds, computed as described above, by $\boldsymbol{b}$. Note that we get (in addition to Theorem 2) a lower bound on $\gamma(w;f)$ if we enclose w by an extra letter λ and compute the lower bound on the number of intersections of $I(\lambda)$, this time without rounding the result up to the next odd integer.

3.2 A dual pair of linear programs

In this section we apply the "Big-M"-method and consider the dual pair of LPs depicted in Figure 3. Recall that A (and thus (A, I)) is totally unimodular, so both LPs have integer solutions. We call an optimal solution $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ of the primal LP R-feasible (R-optimal), if $\boldsymbol{x}^*$ is a feasible (optimal) solution for Problem 2. Note that any $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ with $\boldsymbol{y}^* = \mathbf{0}$ is R-feasible, but not necessarily R-optimal.

Theorem 4. *If $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is an optimal solution of the primal LP with $\boldsymbol{y}^* = \mathbf{0}$, then $x_i^* \in \{0,1\}$ for all i.*

$$\begin{aligned} \mathbf{1}^T\boldsymbol{x} + \boldsymbol{M}^T\boldsymbol{y} &\to \text{min!} \\ A\boldsymbol{x} - I\boldsymbol{y} &= \boldsymbol{b} \\ \boldsymbol{x}, \boldsymbol{y} &\geq \mathbf{0} \end{aligned} \qquad \begin{aligned} \boldsymbol{u}^T\boldsymbol{b} &\to \text{max!} \\ \boldsymbol{u}^T A &\leq \mathbf{1} \\ \boldsymbol{u} &\geq -\boldsymbol{M} \end{aligned}$$

Fig. 3. The primal LP and its dual

If $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is not R-optimal, the lower bound $\boldsymbol{b}$ is not tight. A rough statistic shows that the percentage of such instances increases from less than 5% for $|\Sigma| = 5$ to more than 90% for $|\Sigma| = 25$. For example, both instances shown in Figure 4 have a lower bound of $\boldsymbol{b} = \mathbf{1}$, which is not tight. In such cases we are searching for an adaption of $\boldsymbol{b}$ so that the solution of the primal LP yields an R-optimal $(\boldsymbol{x}^*, \boldsymbol{y}^*)$. Due to recent computational experiments we conjecture the following.

Conjecture 2. Suppose that $\boldsymbol{b}$ is not tight and let $U := \{j : u_j < 0\}$ denote the index set of negative dual variables after the solution of the primal LP. Then there exists $I \subset \Sigma$ so that $I \subset U$, and the primal LP yields an R-optimal solution if we increase b_i by 2 for all $i \in I$.

Figure 1(a) shows an example of an instance for which the primal LP yields an R-feasible solution and an objective value of 4, while we get $\gamma(w; f) = 3$ if we increase the lower bound on the number of intersections of $I(B)$ by 2.

3.3 MaxFlow-MinCut Duality

We get an even stronger duality than the LP duality described in Section 3.2 if we formulate Problem 2 in terms of matroid theory.

Therefore, we identify the set of feasible solutions to Problem 2 with the set of feasible solutions of the equation $A\boldsymbol{x} \equiv \mathbf{1} \bmod 2$ over GF(2). If we replace A by $(A, \mathbf{1})$, this equation is equivalent to $(A, \mathbf{1})\boldsymbol{x} \equiv \mathbf{0} \bmod 2$, if we demand that $x_n = 1$ always. Thus we are seeking a minimal element in the kernel of $(A, \mathbf{1})$, or, in other words, a shortest circuit in the binary vector matroid $(A, \mathbf{1})$, that contains the element $\mathbf{1}$. This yields an equivalent formulation of Problem 2.

Problem 3. Shortest circuit in a clutter

Instance A matrix A with the CO property for rows, where no two sequences of consecutive ones start or end in a common column.

Question Find a shortest circuit in the clutter of all circuits of the binary vector matroid $(A, \mathbf{1})$ that contain the element $\mathbf{1}$.

Note that computing a shortest circuit in a binary matroid is NP-complete in general. Furthermore, recall that A is totally unimodular. Thus $(A, \mathbf{1})$ is a one point extension of a regular matroid. We cite the dual version of Seymour's famous MaxFlow-MinCut theorem.

Theorem 5 (Seymour [4]). *Given a binary matroid $M = (E, \mathcal{I})$ and a specific element $e \in E$, the value of a maximum disjoint packing of cocircuits, each of which contains e, equals the value of a minimum circuit that contains e for all length functions $f : E \to N_{\geq 0}$, if and only if M has no F_7-minor that contains e.*

Now we consider the binary vector matroid $M = (A, \mathbf{1})$ with $e = \mathbf{1}$ and fix the length function to $f \equiv 1$ except for $f(\mathbf{1}) = 0$. Then the length of a shortest circuit that contains $\mathbf{1}$ corresponds to the minimum number of color changes for any instance of Problem 2. Note that we are allowed not only to pack disjoint rows, but also odd sums of symmetric differences of rows.

Theorem 6. *If $(A, \mathbf{1})$ has no F_7-minor that contains $\mathbf{1}$, then the maximal value of a disjoint odd row sum packing equals the minimum number of color changes for any instance $(w; f)$ of Problem 3.*

For example, the instance shown in Figure 4(a) has a value of $\gamma(w; f) = 4$, and $(A, \mathbf{1})$ contains no F_7-minor. A maximal odd row sum packing is given by $\{I(A), I(B), I(C)\}$, $\{I(C), I(D), I(E)\}$, $\{I(B)\}$, and $\{I(E)\}$. The instance in Figure 4(b) does contain an F_7-minor (contract the first column, add $I(E)$ to $I(C)$, and contract the eighth column within $(A, \mathbf{1})$). It has a value of $\gamma(w; f) = 3$, whereas a maximal odd row sum packing consists of only two disjoint odd row sums ($\{I(B)\}$ and $\{I(E)\}$).

(a) (b)

Fig. 4. Instances of Problem 3 without (a) and with (b) an F_7-minor in $(A, \mathbf{1})$

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