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SHORT COMMUNICATION

Polyhedral results on the stable set problem in graphs containing even or odd pairs

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Abstract Even and odd pairs are important tools in the study of perfect graphs and were instrumental in the proof of the Strong Perfect Graph Theorem. We suggest that such pairs impose a lot of structure also in arbitrary, not just perfect graphs. To this end, we show that the presence of even or odd pairs in graphs imply a special structure of the stable set polytope. In fact, we give a polyhedral characterization of even and odd pairs.

Keywords Stable set problem · Stable set polytope · Even pairs · Odd pairs · Integer programming

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1 Introduction

Let $G = (V, E)$ be a simple, undirected graph with $n := |V|$ vertices. A set of vertices $S \subseteq V$ is called a *stable set* in G if no vertices of S are adjacent in G, i.e., $|e \cap S| \leq 1$ for all $e \in E$. We abbreviate $\{v, w\} \in E$ with $vw \in E$.

The *stable set polytope* $\text{STAB}(G)$ is defined as the convex hull of incidence vectors of stable sets in G, i.e.,

$$\text{STAB}(G) := \text{conv}\{x \in \{0, 1\}^n : x_v + x_w \leq 1 \; \forall vw \in E\}.$$

Every incidence vector x of a stable set satisfies the clique inequalities:

$$\text{For every clique } Q \subseteq V : \sum_{v \in Q} x_v \leq 1. \tag{1}$$

Hence, inequalities (1) also hold for every vector x in $\text{STAB}(G)$. The set of non-negative vectors for which Condition (1) holds is known as the *fractional stable set polytope* of G, denoted by $\text{QSTAB}(G)$.

A cycle C of odd length $2k + 1$ for an integer $k > 1$ contains no stable set of size exceeding k and hence, the inequality $\sum_v x_v \leq k$ holds for every x in $\text{STAB}(C)$. Thus, the vector with all components equal to $\frac{1}{2}$ is not in $\text{STAB}(C)$. However, this vector is in $\text{QSTAB}(C)$ as every clique of C has cardinality at most 2. For the complement $\overline{C}$ of such an odd cycle, the inequality $\sum_v x_v \leq 2$ holds for every x in $\text{STAB}(\overline{C})$. Thus, the vector with all components equal to $\frac{1}{k}$ is not in $\text{STAB}(\overline{C})$. However, this vector is in $\text{QSTAB}(\overline{C})$ as every clique of $\overline{C}$ has cardinality at most k.

It follows that, in general, for $\text{STAB}(G)$ to equal $\text{QSTAB}(G)$, the graph G necessarily contains no cycle of odd length at least five and no complement of such a cycle as an induced subgraph. Work of Lovász [5], Fulkerson [4], and Chvátal [3], showed that this condition is satisfied precisely if a graph is perfect, and hence, Berge's Strong Perfect Graph Conjecture was equivalent to the statement that this necessary condition was sufficient (see Chaps. 1 and 2 of [8] for the definition of perfect graphs, statement of the conjecture, and an accessible presentation of the work of Chvátal, Lovász, and Fulkerson). Berge's conjecture was proven by Chudnovsky, Robertson, Seymour, and Thomas in 2006 [1].

An *even (odd) pair* in G is a pair of non-adjacent vertices such that all induced paths between them have even (odd) length.

In 1987, Meyniel [6] proved that a minimal imperfect graph contains no even pair and in 1989, it was conjectured that this is also true for odd pairs [7,9]. Even and odd pairs became an important tool in the study of perfect graphs (see Chap. 4 of [8] for a survey of work in this vein). In particular, even pairs play an important role in a much shortened proof of the Strong Perfect Graph Theorem by Chudnovsky and Seymour [2].

We prove that the presence of even or odd pairs in arbitrary, not just perfect, graphs G implies a particular polyhedral structure of $\text{STAB}(G)$. To this end, we adopt Meyniel's proof technique to prove results about the stable set polytope of an arbitrary graph and an even or odd pair within it.

2 The polyhedral results

Let $G = (V, E)$ be a graph and let $v, w \in V$ be two vertices with $vw \notin E$. We define the graph $G + vw$ as the graph obtained by adding edge vw to G, i.e., $G + vw := (V, E \cup \{vw\})$.

Theorem 1 *Let $G = (V, E)$ be a graph and let $v, w \in V$ with $vw \notin E$. It holds that (v, w) is an odd pair if and only if*

$$\text{STAB}(G + vw) = \{x \in \text{STAB}(G) : x_v + x_w \leq 1\}. \tag{2}$$

Proof For necessity, suppose that Eq. (2) holds and assume that there is an even induced v-w-path in G. Then there is an odd induced cycle containing the edge vw in $G + vw$. The corresponding odd cycle inequality is valid for the stable set polytope $\text{STAB}(G + vw)$, but not for $\{x \in \text{STAB}(G) : x_v + x_w \leq 1\}$; a contradiction.

For sufficiency, suppose that there is no even induced v-w-path in G. Clearly, if x is in $\text{STAB}(G + vw)$, it is in $\{x \in \text{STAB}(G) : x_v + x_w \leq 1\}$. It remains to prove the converse. To this end we want to show that for any x in $\text{STAB}(G)$ with $x_v + x_w \leq 1$ we can find a convex combination of incidence vectors of stable sets of G summing to x, such that each stable set contains only one of $\{v, w\}$.

Clearly, it is enough to show that if S_1 is a stable set containing both v and w and S_2 is a stable set containing neither then we can express $S_1 \cup S_2$ as the union of two stable sets each containing one of $\{v, w\}$. Since there is no even induced v-w-path in G, the connected component K of $G[S_1 \cup S_2]$ containing v does not contain w. Thus, we can use the stable sets $(S_1 - K) \cup (S_2 \cap K)$ and $(S_2 - K) \cup (S_1 \cap K)$. □

For bipartite graphs, successively adding the edges (thus iteratively applying Theorem 1), results in the simplest structure of the stable set polytope: the edge inequalities already suffice. The following generalization of Theorem 1 gives an alternative view on this:

Corollary 1 *Let $G = (V, E)$ be a graph and let $H = (V, E \setminus F)$ with $F \subseteq E$ be a subgraph of G. The edge set $E \setminus F$ contains all odd induced cycles of G if and only if*

$$\text{STAB}(G) = \{x \in \text{STAB}(H) : x_v + x_w \leq 1 \quad \forall vw \in F\}. \tag{3}$$

Proof For necessity, suppose that Eqn. (3) holds and assume that there is an odd induced cycle in G that is not contained in $E \setminus F$. The corresponding odd cycle inequality is valid for the stable set polytope $\text{STAB}(G)$, but not for the polytope given in the right hand side of (3). This is a contradiction to the assumption that Eqn. (3) holds.

For sufficiency, it is easy to see that the statement follows by induction on $|F|$ using Theorem 1. □

As before, let $G = (V, E)$ be a graph and let $v, w \in V$ with $vw \notin E$. We define the graph G/vw as the graph obtained by deleting v and w from G and adding a new vertex vw adjacent to precisely those vertices of G which were adjacent to at least one of v and w in G. We denote the vertex set of G/vw by $V/vw := (V \cup \{vw\}) \setminus \{v, w\}$ and call this operation the contraction of vw in G.

Let $x \in \text{STAB}(G)$ with $x_v = x_w$. We define the contraction of vw in x as the vector $x/vw \in [0,1]^{V/vw}$ with $(x/vw)_u = x_u$ for all $u \in V \setminus \{v, w\}$ and $(x/vw)_{vw} = x_v = x_w$. Analogously to Theorem 1, we state our result on the stable set polytope of graphs containing an even pair:

Theorem 2 *Let $G = (V, E)$ be a graph and let $v, w \in V$ with $vw \notin E$. It holds that (v, w) is an even pair if and only if*

$$\text{STAB}(G/vw) = \{x/vw : x \in \text{STAB}(G),\ x_v = x_w\}.$$

Proof We can prove this statement analogously to Theorem 1 by replacing odd with even pairs as well as replacing the addition of the edge vw by the contraction of vw in the proof of Theorem 1. Accordingly, the role of stable sets containing only one of $\{v, w\}$ and the role of stable sets containing both or neither of $\{v, w\}$ is inverted. □

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