Political Districting Problem: Literature Review and Discussion with regard to Federal Elections in Germany

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Abstract. Electoral districts have great significance for many democratic parliamentary elections. Voters of each district elect a number of representatives into parliament. The districts form a partition of the electoral territory, meaning each part of the territory and population is represented. The problem of partitioning a territory into a given number of electoral districts, meeting various criteria specified by laws, is known as the Political Districting Problem. In this paper, we review solution approaches proposed in the literature and survey districting software, which provides assistance with interactive districting by hand or even decision support in the form of optimization-based automated districting. As a specific application, we consider the Political Districting Problem for the federal elections in Germany. Regarding the present requirements and objectives, we discuss and examine the applicability of the approaches mentioned in the literature to this specific German Political Districting Problem.

Keywords: Redistricting; Electoral District Design; Solution Approaches; Literature Survey; (Re)Districting Software; OR in Government

1 Introduction

In preparation for an upcoming parliamentary election, a country is generally subdivided into *electoral districts*. These districts are of fundamental importance in democratic elections, because the voters of each district elect a number of representatives into parliament. In general, the number of seats staffed by an electoral district is determined a priori in line with the district's population. In many cases, exactly one seat is assigned to each electoral district. This calls for a balance in population distribution among the districts. Owing to population changes, the partition into electoral districts, i.e., the *districting plan*, needs regular adjustments.

The *Political Districting Problem* (PDP) denotes the task of partitioning a geographical territory, such as a country, into a given number of electoral districts while considering different constraints and (optimization) criteria. Every country has its own electoral system and laws. Therefore, the legal requirements and their particular importance for a districting plan differ across application cases.

Models and solution approaches proposed in the literature are primarily addressed to the PDP in the United States of America. The particular motivation is mostly to tackle the suspicion of applying *gerrymandering*. Gerrymandering is the practice of creating (dis)advantages from the territorial subdivision for a certain political party, a candidate, or a social class in order to gain or lose seats. The term "gerrymandering" dates back to the early 1800s when Elbridge Gerry, the acting governor of Massachusetts, signed a bill that redistricted the state to benefit his Democratic-Republican Party. A cartoonist[3] realized that one of the new districts resembles the shape of a salamander. As a blend of the word salamander and Governor Gerry's last name, the "Gerry-Mander" was coined [Griffith, 1907]. Basically, gerrymandering can be utilized in pure majority voting systems (first-past-the-post systems). By contrast, pure proportional representation precludes gerrymandering. The symptoms of manipulating geographic political boundaries are usually odd-shaped districts, such as the original gerrymander from 1812. For deeper insights into the topic of gerrymandering, see [Cox and Katz, 2002] and [McGann et al., 2016].

Today, we have to deal with "the *digital gerrymander*," as Berghel [2016] recently stated. Nowadays, computers and mathematics are exploited in an arms race between subtly performing and objectively identifying gerrymandering. Mathematical models and algorithms are transparent as they are defined in a precise way. However, they are only unbiased as long as they are not fed with political or social data.[4]

One answer to the highly discussed malpractice of gerrymandering is the *compactness* of electoral districts. Odd-shaped districts are undesirable, because

[3] The first known use of the word "gerrymandering" appeared in "The Gerry-Mander: A new species of Monster which appeared in Essex South District in Jan. 1812", *Boston Gazette*, March 26, 1812. The article is available at `http://www.masshist.org/database/1765` (visited on Oct 1, 2018).

[4] Former US president Ronald Reagan is cited in [Altman, 1997]: "There is only one way to do reapportionment – feed into the computer all the factors except political registration."

this might be an indication for gerrymandering. The more circle-like or square-like an area is shaped, and the less elongated and frayed it is, the more compact it is. However, there is no uniform definition of compactness and its measurement, neither in the literature nor in court decisions. Horn et al. [1993] lists over 30 compactness indicators. For detailed discussions about compactness, see [Young, 1988], [Niemi et al., 1990], [Chambers and Miller, 2010], and [Fryer and Holden, 2011].

Of late, another proposed measure of gerrymandering has gained (public) attention. The Supreme Court of the United States of America considers the *efficiency gap* in a partisan gerrymandering case in Wisconsin.[5] The efficiency gap captures the difference in "wasted votes" between two parties engaged in an election. See [McGhee, 2014] and [Stephanopoulos and McGhee, 2015] for more details and the calculation of the efficiency gap in a hypothetical election scenario.

Besides compactness, the following two criteria are mostly considered in the literature of PDP: *Contiguity*: Each electoral district has to be geographically contiguous. *Population balance*: In order to comply with the principle of electoral equality, i.e., one person-one vote, the differences in population among the electoral districts have to be preferably small. In practice, the law defines a limit on the deviation.

One specific application, which is only partly addressed in the literature is the PDP for the German parliamentary elections: the *German Political Districting Problem* (GPDP). Since Germany's electoral system is a mixture of proportional representation and uninominal voting in the electoral districts, the effect of applying gerrymandering is comparatively small. However, the design of the electoral districts is frequently called into question by the German public, too. Additionally, the European organization OSCE [2009, 2013] officially criticized the German districting plan regarding its large population imbalance. Referring to the Code of Good Practice in Electoral Matters of the Venice Commission [2002], it is pointed out that the deviations of district population are way too large in Germany.

The PDP is a special *districting problem*, *territory design problem*, or *zone design problem*. This kind of problem has been applied to an extensive number of fields. Within this survey, we disregard all works not specifically addressing

[5] *Gill v. Whitford*, United States Supreme Court case, No. 15-cv-421-bbc, 2016 WL 6837229 (E.D. Wis. Nov. 21, 2016), docket no. 16-1161.

the PDP. A broad review of different districting applications is given by Kalcsics et al. [2005]. Moreover, Kalcsics et al. [2005] provides one of few papers that consider the districting problem independently from a concrete practical background.

Contribution. In this article, we review solution approaches, models, and algorithms proposed in the literature for the PDP. The considered constraints and optimization criteria differ across applications. Besides a general literature survey, we specifically consider the legal requirements and principles given for the delimitation of electoral districts for federal elections in Germany. In addition to the review of solution approaches and a suitability evaluation for the German case, we survey districting software that offers either assistance with manually districting or decision support in the form of optimization-based automated districting. Unfortunately, most software is only commercially available and promising open source projects are outdated.

If a reader is not interested in the specific German application but in the general literature review of the solution approaches for the PDP and districting software, one can skip Sections 3 and 5.

Outline. In Section 2, we present a definition of the PDP and provide a unified mathematical model. We discuss extensions and comment on the problem's computational complexity. In Section 3, we introduce the basics of the German electoral system, comment on specifics, and define the GPDP on the basis of presented legal requirements. In Section 4, we review the literature's solution approaches as well as available (re)districting software for PDP. We discuss the approaches' applicability to the considered German problem in Section 5. The paper closes with a conclusion in Section 6.

2 Political Districting Problem

A territory, e.g., a country or federal state, has to be partitioned into $k \in \mathbb{N}$ electoral districts meeting certain (legal) criteria. For this purpose, a discretization of the territory is given in the form of a partition into $n \in \mathbb{N}$, $n \gg k$ geographical units. These units can be, e.g., municipal associations, municipalities, city districts, or census tracts. Most PDP models assume that each unit has to be assigned to exactly one electoral district, i.e., a unit can not be split. This assumption is not a relevant restriction for applications in practice, as a main requirement is not to split up existing administrative units like municipalities or city districts. We follow this assumption in our modeling.

After the introduction of a population graph in Section 2.1, a basic definition of the PDP is given in Section 2.2. In Section 2.3, the computational complexity of the PDP is analyzed.

2.1 Population Graph

To model PDP, it is a widely spread and quite natural idea to use a connected graph $G = (V, E)$ representing adjacencies. In the so-called *population graph* (or *contiguity graph*) G, a *node* $i \in V$ represents a geographical unit. Each node $i \in V$ is weighted with its population $p_i \in \mathbb{N}$. It is common to call V the set of *population units*. An undirected *edge* $(i, j) \in E$ with nodes $i, j \in V$ exists if and only if the corresponding areas share a common border. Depending on the given criteria, further parameters for the nodes and edges may be given. See Figure 1 for an exemplar population graph and its construction based on a given discretization of the territory.

Fig. 1. Constructing a population graph: population units as nodes, edges represent adjacent units (administrative boundaries: © GeoBasis-DE / BKG 2016).

2.2 Mathematical Model

Based on a given population graph $G = (V, E)$ and a number of electoral districts $k \in \mathbb{N}$, we give a basic definition of the PDP. It can be extended with further criteria and requirements.

The task is to find a districting plan $\mathcal{D}$, i.e., a partition of the set of population units V in electoral districts

$$\mathcal{D} = \{D_1, D_2, \ldots, D_k\} \text{ with disjoint } D_\ell \subseteq V\ \forall \ell \text{ and } \bigcup_\ell D_\ell = V. \tag{1}$$

The basic PDP calls for electoral districts D_ℓ with contiguity and population balance. Continuity leads to the constraint

$$G[D_\ell] \text{ connected} \qquad \forall \ell \in \{1, \ldots, k\}, \tag{2}$$

where graph $G[D_\ell] := (D_\ell, E(D_\ell))$ with set of edges $E(D_\ell) := \{(i,j) \in E : i, j \in D_\ell\}$ is the subgraph of $G = (V, E)$ induced by node set $D_\ell \subseteq V$.

Population balance can be aimed for in the objective function or, as stated in the following, implemented as a range constraint limiting the amount of legal imbalance. Let $\bar{p}$ be the *average population of an electoral district.* As per definition, a district D_ℓ with $\sum_{i \in D_\ell} p_i = \bar{p}$ has perfect population balance. In most applications $\bar{p} = \frac{\sum_{i \in V} p_i}{k}$ holds.[6] For given bounds $\check{p}$, $\hat{p}$ with $\check{p} \leq \bar{p} \leq \hat{p}$ the districting plan $\mathcal{D}$ has to fulfill the range constraint of population balance

$$\check{p} \leq \sum_{i \in D_\ell} p_i \leq \hat{p} \qquad \forall \ell \in \{1, \ldots, k\}. \tag{3}$$

The basic PDP (1)–(3) can be extended by further criteria that are implemented in the form of an objective function or (range) constraints. The multiplicity of relevant criteria is extensively discussed in [Williams, 1995], [di Cortona et al., 1999, Chapter 10], [Kalcsics et al., 2005], and [Webster, 2013]. Let c be a criterion, e.g. compactness. Let $c(\mathcal{D})$ and $c(D_\ell)$ be indicators that measure the criterion for a districting plan $\mathcal{D}$ and an electoral district D_ℓ, respectively. Note that the measurement of most criteria, e.g., compactness, is not clearly given by the legal requirements and is subject to discussion. The basic PDP is extended with criterion c by adding objective

$$\text{maximize / minimize} \qquad c(\mathcal{D}) \tag{4}$$

[6] This equation does not hold for the German case in general (cf. Section 3): The GPDP decomposes into 16 independently solvable PDPs, each with the same $\bar{p}$ specified by the entire GPDP instance and not by the individual subproblem.

or adding district sharp range constraints with given bounds $\check{c}, \hat{c}$

$$\check{c} \leq c(D_\ell) \leq \hat{c} \qquad \forall \ell \in \{1, \ldots, k\}. \tag{5}$$

Range constraints $\check{c} \leq c(\mathcal{D}) \leq \hat{c}$ regarding the entire districting plan $\mathcal{D}$ are possible as well. Implementing more than one criterion as objective leads to a multi-criteria optimization problem.

2.3 Complexity

PDP (1) – (3) with its two basic criteria, contiguity and population balance, is equivalent to the following combinatorial task: Partition a node-weighted graph into a given number of connected and weight-restricted subgraphs. On paths and trees this problem can be solved in linear time [Lucertini et al., 1993] and polynomial time [Ito et al., 2012], respectively. For series-parallel graphs this problem gets NP-hard [Ito et al., 2006]. Thus, the PDP is NP-hard in general.

Minimizing population imbalance $\sum_{\ell=1}^{k} |\bar{p} - \sum_{i \in D_\ell} p_i|$ in the objective of the PDP instead of limiting it with constraints (3) leads to an NP-hard optimization problem even on trees [De Simone et al., 1990].

The most frequently cited work in the context of the PDP's complexity is [Altman, 1997]. Among other things, the author analyzes that computing a districting plan with maximally compact electoral districts is NP-hard. Thereby, population units are given as points in the plane and the considered decision problem asks if these points can be covered by k discs of a certain diameter (cf. [Johnson, 1982]). Connectivity conditions are neglected.

3 German Political Districting Problem

In Germany, the effect of applying gerrymandering is comparatively small, because an electoral system with mixed-member proportional representation is applied. Although German electoral districts are regularly revised and discussed.[7] Continually and even from an official authority, the very liberal and practically

[7] (i) 2002 German federal election: *Bundestagswahl 2002 - Die umstrittenen Wahlkreise*, S. Eisel and J. Graf, Konrad-Adenauer-Stiftung e.V., Jan. 2002.
(ii) 2017 North Rhine-Westphalia state election: *Im Essener Süden ist die SPD jetzt klar im Vorteil*, WAZ, online, 06/11/2015.
(iii) 2018 Hessian state election: *Beuthe-Wahlkreise*, Frankfurter Rundschau, online, 12/11/2017.

exploited deviation limits for a district's population are criticized [OSCE, 2009, 2013].

In Section 3.1, the basic elements of the German electoral system including the role of electoral districts is introduced. More details are given in the Federal Election Act (German: Bundeswahlgesetz, abbreviated to BWG, cf. [Schreiber et al., 2017]) and on the website of the German Federal Returning Officer [n.d., online]. In Section 3.2, the German legal requirements for electoral districts are presented in detail. Based on that and the basic PDP (cf. Section 2.2), the German Political Districting Problem (GPDP) is defined in Section 3.3. Its problem size is analyzed in Section 3.4.

3.1 Electoral System of Germany and the Role of Electoral Districts

In German federal elections, voters elect the members of the national parliament, which is called *Bundestag*. The Bundestag can be compared to the lower house of parliament, such as the House of Commons of the United Kingdom or the United States House of Representatives. The German election system is that of a so-called *personalized proportional representation*, i.e., proportional representation in combination with a candidate-centered first-past-the-post system in the electoral districts.

Every German voter has two votes. With the first one, voters select their favorite candidate to represent their electoral district in the parliament. Parties may nominate electoral district candidates, but independent candidates are also possible. Every candidate who wins one of the 299 electoral districts is guaranteed a seat. Approximately half the seats in the Bundestag are assigned by these *direct mandates*. The second vote is given to a party. The result of these votes determines the relative strengths of the parties represented in the Bundestag. This, together with the fact that every district winner has a seat for certain, forms the root of a major weakness in the German electoral system — the inability to determine the size of the parliament in advance. This is explained in the following.

From the legally prescribed total of 598 ($= 2 \cdot 299$) seats, the number of seats each party is entitled to is determined on the basis of the result of the second votes. Whenever a party won more direct mandates than it was entitled to by its share of second votes, the so-called *overhang mandates* arose. In other words, overhang mandates are direct mandates not covered by second votes. To maintain proportionality, which is given by the distribution of second votes, additional *balance mandates* for otherwise underrepresented parties are created. This leads to new seats exceeding the initially targeted total of 598. Thus, the size of the Bundestag depends on the outcome of the elections and is theoretically unbounded.

In the 2017 election, the described weakness led to a parliamentary size of historic dimension. The election yielded the largest Bundestag ever and, simultaneously, the largest democratically elected national parliament in the world. A total of 46 overhang mandates led to 65 additional balance mandates – the resulting Bundestag had 709 members instead of 598 as planned. This fact highlights the need for a reform. In order to limit growth in the number of seats, (political) scientists discuss to change the number of electoral districts in Germany [Behnke et al., 2017; Grotz and Vehrkamp, 2017; Pukelsheim, 2018]. This implies numerous carefully considered adjustments to the districting plan. Hence, in Germany the PDP is more relevant than ever before, and suitable solution methods must definitely be part of current discussions.

3.2 Legal Requirements and Criteria for German Electoral Districts

The essential legal basis of electoral districts and their delimitation for German federal elections is documented in the Federal Election Act (BWG).[8] Those legal requirements have been complemented by the German Constitutional Court (German: Bundesverfassungsgericht, abbreviated to BVerfGE).[9] In Germany, the *number of electoral districts* $k \in \mathbb{N}$ stands at 299. In no particular order, the following principles shall be observed when partitioning Germany into electoral districts.

(a) Decomposability into 16 subproblems. Germany comprises 16 federal states (German: Bundesländer, cf. Table 1), denoted by the set S. The constitutional principle of federalism implies that electoral districts have to respect the federal states' boundaries. The number of electoral districts is apportioned among the states $s \in S$ by means of the *divisor method with standard rounding.* For more insights into apportionment methods, see [Balinski and Young, 1982] and [Pukelsheim, 2017]. We denote the *number of electoral districts of state* $s \in S$ with $k(s) \in \mathbb{N}$, $k(s) \geq 1$. Of course, $\sum_{s \in S} k(s) = k$ holds. Overall, the GPDP can be subdivided into 16 independently solvable PDPs – one for each federal state.

[8] Cf. section 3, subsection 1 BWG.
[9] Cf. BVerfGE 95, 335 in 1997, BVerfGE 121, 226 in 2008, BVerfGE 130, 212 in 2012.

(b) Population balance. In order to comply with the principle of electoral equality, which is anchored in the German constitution, every electoral district must preferably comprise the same number of people. The law defines a two-staged deviation scope: A *tolerance limit*, stating that a deviation from the average district population should not exceed 15%. If the deviation is greater than 25% (*maximum limit*), the appropriate district's boundaries shall be redrawn. In determining population figures, only German people are considered.

(c) Contiguity. Each electoral district should form a continuous area.

(d) Conformity to administrative boundaries. Where possible, the boundaries of administrative subdivisions should be respected. This criterion supports conformity between the boundaries of electoral districts and already existing official and rooted regions, i.e., municipalities, and rural and urban districts.

(e) Continuity. Between two consecutive elections, the adjustments of the electoral districts should be as small as possible. The aim is to achieve the greatest possible continuity in the districting plan.

3.3 Definition of German Political Districting Problem

Based on the legal requirements presented in Section 3.2, we distinguish between *hard* and *soft requirements* corresponding to the GPDP's constraints and objectives, respectively.

Decomposability into 16 subproblems (a), maximum population deviation limit in (b), and contiguity (c) are *hard constraints*. All remaining requirements are *soft constraints*: tolerance population limit in (b), administrative conformity (d), and continuity (e). We model the GPDP as 16 independently solvable multi-objective PDPs. Every individual soft constraint, i.e., objective criterion, influences others. For example, improving the conformity to administrative boundaries may need adjustments to the districts which is in contrast to the criterion of continuity. Officially, there is no explicit order or trade-off between the objective criteria in law nor court resolutions. Goderbauer and Wicke [2017] analyzed the districting plans of the 2013 and 2017 German elections in detail, and deduced the following descending order of importance for the objective criteria in practice: (e) continuity, (d) administrative conformity, and (b) tolerance population limit.

Given a suitable population graph $G = (V, E)$ of Germany, number of electoral districts $k(s) \in \mathbb{N}, k(s) \geq 1$ for each state $s \in S$ with $k := 299 = \sum_{s \in S} k(s)$, and average district population $\bar{p} := \frac{\sum_{i \in V} p_i}{k}$. The 16 German federal states $s \in S$ partition the set of population units $V = \bigcup_{s \in S} V_s$. For each state $s \in S$ a population graph $G_s := (V_s, E_s) := G[V_s]$ arises. Solving the GPDP is equivalent to solving the following PDP (cf. Section 2.2) for each $s \in S$.

Find

$$\mathcal{D}_s = \{D_1, \ldots, D_{k(s)}\} \text{ with disjoint } D_\ell \subseteq V_s \; \forall \ell \text{ and } \bigcup_\ell D_\ell = V_s \tag{6}$$

so that

$$G_s[D_\ell] \text{ connected} \qquad \forall \ell \in \{1, \ldots, k(s)\} \tag{7}$$

$$0.75\,\bar{p} \leq \sum_{i \in D_\ell} p_i \leq 1.25\,\bar{p} \qquad \forall \ell \in \{1, \ldots, k(s)\} \tag{8}$$

while

max continuity to the previous election's districts (9)

max conformity between elect. districts and adm. boundaries (10)

max number of districts complying with 15% tolerance limit (11)

min amount of deviations between district population and $\bar{p}$ (12)

The union $\mathcal{D} := \bigcup_{s \in S} \mathcal{D}_s$ describes a *districting plan* for the GPDP. Objective criteria (9) and (10) refer to the most important soft constraints (e) and (d), respectively. The tolerance limit of population balance and the population balance (b) itself are implemented by objective criteria (11) and (12), respectively.

German law provides no measurement of these criteria. We deliberately omit to cast (9)–(12) in mathematical terms. Determining suitable measurement functions for especially the two most important objectives in German practice, continuity and administrative conformity, does not seem to be a straight-forward task. We additionally elaborate the literature review in this work to record suitable measurements for the GPDP's objectives.

With regard to administrative conformity, Goderbauer and Wicke [2017] point out that, in the German case, this objective deals with at least the following hierarchical divisions (cf. Figure 2): municipalities, municipal associations, rural and urban districts, and governmental regions. The rural and urban districts are most comparable in population numbers to an electoral district. On the one hand, there are electoral districts that contain several urban/rural districts completely. On the other hand, some urban/rural districts are divided into multiple electoral districts. Apart from large cities, municipalities and municipal associations are usually too small to form an electoral district. Governmental districts comprise several electoral districts. A measurement for administrative conformity has to consider these characteristics.

3.4 Size of German Political Districting Problem

As mentioned, the GPDP decomposes into 16 independently solvable PDPs. Table 1 gives an overview of the sizes of the PDPs. The column entitled Gem (=Gemeinden in German) indicates the number of municipalities, giving an impression of the order of magnitude of population units in the population graphs. Since there are German cities (being in particular municipalities) with a population greater than the maximum population limit $1.25\,\bar{p}$, these cities have to

	federal state	German population	$k(s)$	number of units at administrative level			
				RB	Kr	VB	Gem
01	Schleswig-Holstein	2 680 368	11	1	15	173	1 112
02	Hamburg	1 521 536	6	1	1	1	1
03	Niedersachsen	7 292 572	30	1	46	434	998
04	Bremen	569 478	2	1	2	2	2
05	Nordhein-Westfalen	15 758 084	64	5	53	396	396
06	Hessen	5 293 234	22	3	26	430	430
07	Rheinland-Pfalz	3 671 099	15	1	36	192	2 306
08	Baden-Württemberg	9 372 306	38	4	44	462	1 103
09	Bayern	11 372 546	46	7	96	1 426	2 099
10	Saarland	905 965	4	1	6	52	52
11	Berlin	2 972 331	12	1	1	1	1
12	Brandenburg	2 395 418	10	1	18	200	417
13	Mecklenburg-Vorpommern	1 553 846	6	1	8	116	754
14	Sachsen	3 926 810	16	1	13	312	426
15	Sachsen-Anhalt	2 160 479	9	1	14	122	218
16	Thüringen	2 090 264	8	1	23	219	849
	Germany	73 536 336	299	31	402	4 538	11 164

Table 1. German population, number of electoral districts $k(s)$ of federal state $s \in S$ at federal elections in 2017, number of units at different administrative levels. German population as of 2015/09/30, based on Census 2011 and number of units at different administrative levels as of 2016/09/30 (© Statistisches Bundesamt, Wiesbaden, 2016). See Fig. 2 for used acronyms in last four columns.

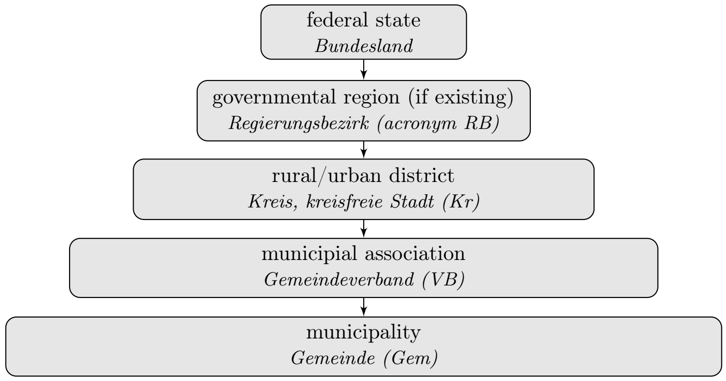

Fig. 2. Hierarchical administrative divisions in Germany.

be divided at least on the level of their boroughs to facilitate a feasible districting plan. Since the GPDP is defined on the basis of indivisible population units (cf. Eq. (6)), this leads to more population units than municipalities. As has been pointed out already, the conformity between electoral districts and administrative boundaries is an important objective and involves several levels of administrative units, e.g., rural and urban districts, municipal associations. For orientation purposes, Table 1 provides the numbers of units at different administrative levels. The administrative divisions, along with their acronyms used in Table 1, are given in Figure 2. See [Goderbauer, 2016] for illustrations of a municipality-level population graph for each German federal state and information about the number of edges in these graphs.

4 Literature Review: Solution Approaches and Software

In this survey, we focus on work proposing solution approaches with explicit reference to the PDP by mentioning keywords such as *political (re)districting*, *non-partisan districting*, or *electoral district design*. This leads us to a set of 49 publications. Each of these publications is represented by a point in Figure 3, indicating its year of publication and the number of citations. Do note that some points overlap each other. In the next sections, we restrict our attention to the 28 black, labeled publications. These curated papers provide pioneering or ground-breaking results; mainly recent ones offer promising new approaches. The 21 remaining publications (grey dots) are not discussed further in this overview, as they tend to contribute to applications rather than methodology. They mostly

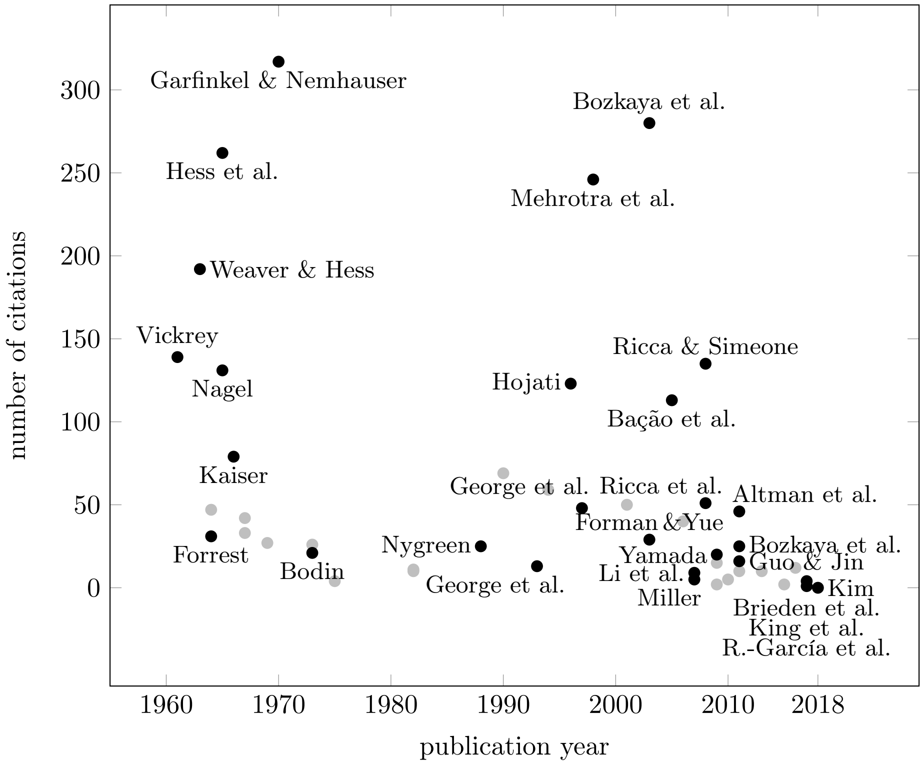

Fig. 3. Publications on PDP, its year of publication, and number of citations (source of number of citations: Google Scholar as of Oct. 6, 2018).

take up the work of the discussed PDP papers or propose methods and models with only little modifications to previous (PDP) results.[10] When separating the grey papers, we ensure that they do not contain any contributions to the measurement of the GPDP criteria. The gray publications are not cited in the next sections but listed in the "Further Reading" bibliography at the end of this paper.

Other literature reviews on the PDP are [Papayanopoulos, 1973], [Williams, 1995], [di Cortona et al., 1999, Chapter 12], and [Ricca et al., 2011].

In the following Section 4.1, the PDP literature and its solution approaches are discussed. In Section 4.2, software tools for redistricting are presented.

[10] An exception to this is the work of Chou and Li [2006] (grey dot, 40 citations). The authors carry out a simulation using a q-state Potts model that has been in use in statistical physics since the 1950s but has not yet been mentioned in connection with the PDP.

4.1 Solution Approaches for PDP in Literature

Exact Methods. Since the PDP is NP-hard (cf. Section 2.3), most approaches are heuristics and assure appropriate computational effort. Nevertheless, there are some *exact methods* for solving the PDP. Garfinkel and Nemhauser [1970] presented a two-phase algorithm and solved instances of up to 40 population units and 7 districts in a reasonable amount of time. After generating *all* feasible electoral districts, a set partitioning model was used to provide a districting plan. This implicit enumeration approach was not sufficient for solving large-scale instances. [Garfinkel and Nemhauser, 1970] is the most cited publication in the surveyed literature of the PDP (cf. Figure 3).

An algorithm comparable with the work of Garfinkel and Nemhauser was presented by Nygreen [1988]. Using implicit enumeration and a set partitioning problem, the author grouped 38 parliamentary districts of Wales together into 4 European electoral districts. In the conclusions of the paper, the author noted that the equivalent PDP for England (with ≥ 500 parliamentary districts, ≥ 60 European electoral districts) would be too large for the approach to terminate in reasonable computation time.

Li et al. [2007] used a quadratic programming model to redistrict New York. The model's decision variables are continuous, denoting the percentage of assigning a population unit to an electoral district. The authors thus assumed to be able to split population units at any position. This is contrary to our definition of the PDP given in Section 2.2.

Kim [2018] applied a contiguity model proposed by Williams [2002a,b] to solve PDPs on artificial grid instances. Assuming planarity of the used graph, Williams [2002b] developed a remarkably small and strong mixed-integer programming model that ensures connectivity of node-induced subgraphs. However, Validi and Buchanan [2018] have shown, that the formulation of Williams is incorrect. Fortunately, the same authors provide a simple fix. Based on this, the work of Kim [2018] needs to be revised.

Exact/Heuristic: Column Generation. Since the already mentioned enumeration approach of Garfinkel and Nemhauser [1970] is not suitable to deal with larger instances, Mehrotra et al. [1998] evolved the idea into a *column generation/branch and price* procedure. They considered more criteria and got faster results, without reducing the quality of the obtained solutions in any significant way. The procedure generated suitable electoral districts iteratively in the subproblem of a column generation approach. In fact, districts are required to be subtrees of shortest path trees [Zoltners and Sinha, 1983] which induces connectedness and compactness. The master problem of the column generation approach is a set partitioning problem. In this problem, k districts are selected out of the set of already generated feasible districts. In general, the technique of column generation and of branch and price can be used to solve optimization problems exactly [Lübbecke and Desrosiers, 2005]. Even so, the algorithm of Mehrotra et al. [1998] remains a heuristic, since some contiguous but most likely irrelevant districts are excluded due to the contiguity model used.

Heuristic: Greedy. Probably the first *heuristic approach* for the PDP was a *multi-kernel growth method* introduced by Vickrey [1961]. Vickrey's publication in a political journal contained a quite rudimentary description of a *greedy algorithm*. Bodin [1973], who presented another multi-kernel procedure, was one of the first to mathematically introduce the concept of a population graph.

The main steps of multi-kernel growth methods are illustrated in Figure 4. First, the centers of the districts must either be given or found by a preprocessing step (Fig. 4, left). Next, the districts grow from their respective centers by adding neighboring units according to a chosen algorithm (Fig. 4, middle). The procedure stops when every unit is assigned to one district, hopefully producing a feasible districting plan (Fig. 4, right). Although, multi-kernel growth methods are fast, they usually generate districting plans with a low population balance as well as a low compactness factor due to left-over population units during

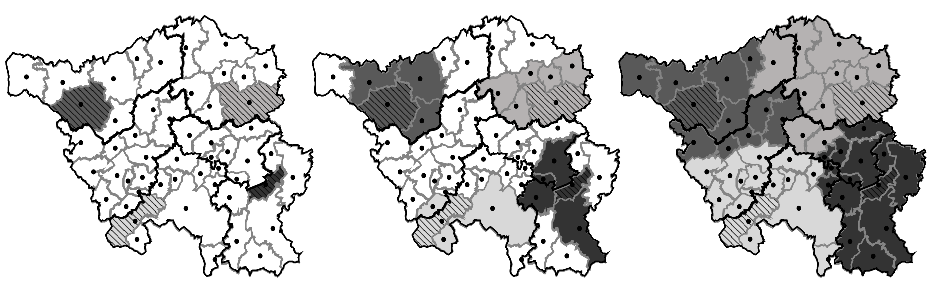

Fig. 4. Greedy heuristic (boundaries: © GeoBasis-DE / BKG 2016):
Left: Every district has a given starting point (crosshatched areas).
Middle: Add neighbouring population units to the districts.
Right: Stop when every unit is assigned to one district.

the growth process. Therefore, a postprocessing step is necessary to produce satisfying results.

Heuristic: Location-Allocation. Weaver and Hess [1963] pioneered in applying a *location-allocation* approach to solve the PDP. In a second paper, they formalized their work [Hess et al., 1965]. In several publications, other authors used their model as a basis.

This kind of method consists of repeating location and allocation steps until the assignment of units to districts does not change anymore. As shown in Figure 5, a location-allocation step takes an assignment of units to districts as input (Fig. 5, left). Thereafter, the centers of the current districts are located according to some measurements (Fig. 5, middle). The output is a new mapping from each unit to its nearest new center (Fig. 5, right). Afterward, this new assignment is used as an input for the next iteration. To ensure population balance, some models allow assigning population units to more than one district, e.g., with a certain percentage. To resolve those splits, a second algorithm is implemented. All in all, these location-allocation methods can not ensure producing connected districts.

George et al. [1993, 1997] expanded the location-allocation approach of Hess et al. [1965] by solving a minimum cost network flow problem. In their network, population units are assigned to new district centers in the following manner. Each population unit i is represented as a node with supply p_i, its population. Each electoral district is represented as a node with no demand or supply, and all electoral district nodes are connected to a super sink node with demand $\sum_i p_i$. Flow from every population unit to the super sink is possible through

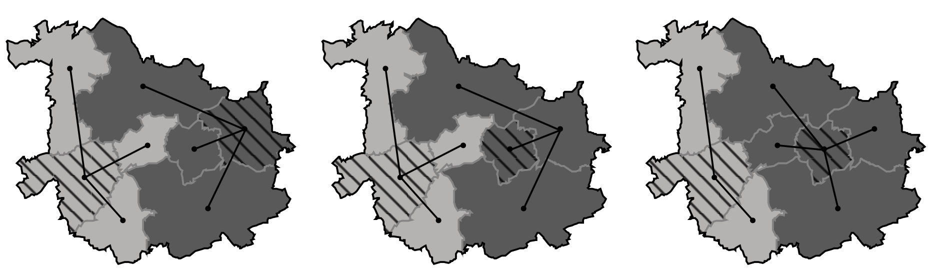

Fig. 5. Location-allocation step/heuristic (boundaries: © GeoBasis-DE / BKG 2016):
Left: Allocate points to nearest (given) center.
Middle: Locate new centers of the districts.
Right: Allocate points to nearest new center.

each electoral district. With respect to flow balance equation and nonnegativity constraints, a minimum cost flow is computed and determines how population units are allocated to electoral districts. The authors point out several options to choose the arc costs in that network and to consider various types of criteria. Population units that are allocated to more than one electoral district, i.e., splits, are reassigned solely to the district with the highest proportion of population for that unit.

Hojati [1996] used a Lagrangian relaxation method from the general location-allocation literature to find the district centers and resolved the occurring splits using a sequence of capacitated transportation problems.

Heuristic: Local Search. Nagel [1965] and Kaiser [1966] solved the PDP by transferring and swapping population units between neighboring electoral districts, as described in Figures 6 and 7. The candidate districts involved in a swap/transfer are chosen according to some criteria such as size and compactness (Fig. 6 and 7, left). Units to swap/transfer are determined using an objective function calculating the benefits of the resulting solution (Fig. 6 and 7, middle). Population units with a best score are swapped/transferred (Fig. 6 and 7, right). Once again, the algorithm stops when no improving candidates can be found or a stop criterion is reached. The swap/transfer method can be seen as an early approach to the modern *local search heuristics*.

Bozkaya et al. [2003] proposed a tabu search algorithm considering a group of criteria in the objective function. The algorithm is enhanced with an adaptive memory procedure [Rochat and Taillard, 1995] that constantly combines districts of good solutions to construct other high quality districting plans. This concept is also known in the field of genetic algorithms. In [Bozkaya et al., 2011], the same authors report on their successful implementation of new electoral districts for the city council elections in Edmonton, Canada.

Yamada [2009] formulated the PDP as a minimax spanning forest problem and presented two local search algorithms operating on trees on the population graph. Owing to the tree model, the algorithms guarantee contiguity of the obtained districts.

Ricca and Simeone [2008] applied several local search variations to the PDP and compared their respective performance in a case study. They determined advantages and disadvantages of these methods.

King et al. [2017] improved local search approaches for the PDP by proposing a procedure which substantially reduces computations needed for the connectiv-

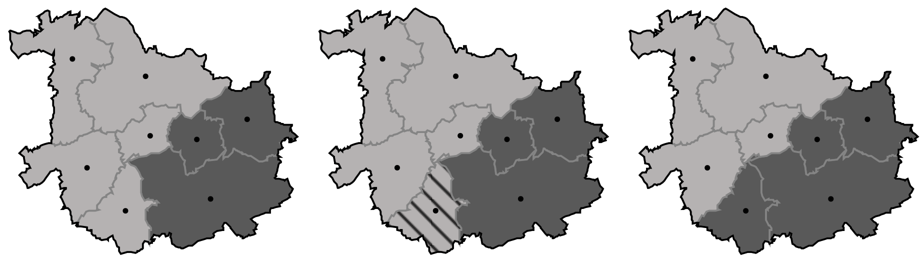

Fig. 6. Transfer step, local search heuristic (boundaries: © GeoBasis-DE / BKG 2016):
Left: Choose a "donor" (light gray) and "receiver" district (dark gray).
Middle: Find best unit to transfer.
Right: The chosen unit is now assigned to the receiver district.

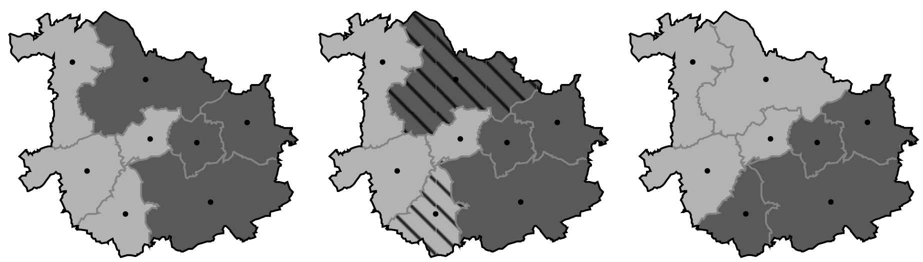

Fig. 7. Swap step, local search heuristic (boundaries: © GeoBasis-DE / BKG 2016):
Left: Choose two districts that will swap a population unit.
Middle: Find best units to swap.
Right: Swap the chosen units between the two districts.

ity check. They use a framework called geo-graph [King et al., 2015, 2012]. Applying this concept decreases the contiguity-related computations by at least three orders of magnitude compared to simple graph search algorithms like breadth-first search and depth-first search as used by, e.g., Ricca and Simeone [2008]. To apply the geo-graph model, assumptions are made concerning the population units, especially the geometry of the units' boundaries. Forbidden are: (i) units whose area is fully nested inside the area of another unit and (ii) units with several non-contiguous areas. King et al. [2017] proposed preprocessing methods to eliminate violations of these assumptions. To evaluate the performance of the geo-graph model, a simple steepest descent local search algorithm is implemented. The authors were able to handle instances with up to 340 000 population units and 29 electoral districts.

Heuristic: Nature-inspired and Probabilistic Algorithms. Forman and Yue [2003] proposed a *genetic algorithm* to solve the PDP. Their work is based on existing genetic algorithms for the traveling salesman problem [Larranaga et al., 1999]. Bação et al. [2005] picked up on the same idea, although they decided to use a clustering heuristic as a basis for their procedure. In a comparative study, Rincón-García et al. [2017] analyzed the performance of four different *nature-inspired and probabilistic metaheuristics* for PDP: simulated annealing, particle swarm optimization, artificial bee colony, and a method of musical composition.

Heuristic: Geometric. As the PDP asks for a partition of the plane into districts, it seems reasonable to apply methods from the field of *computational geometry*. Forrest [1964] was the first to work on this for the PDP. Unfortunately, no explicit algorithm or computational results are given for the proposed method of diminishing halves. Other authors took up the idea and developed methods based on the concept of *Voronoi diagrams* [Aurenhammer and Klein, 2000; Okabe et al., 2009]. Voronoi regions are inherently compact and contiguous, which is why they are often named in the context of striving against gerrymandering.

Miller [2007] applied an algorithm for (centroidal) Voronoi diagrams on data of the US state Washington. As the author puts no population constraints on the Voronoi diagram, the method creates districts with bad population balance.

In contrast to Miller, who considered the territory as a continuous area, Ricca et al. [2008] proposed a Voronoi heuristic for the PDP on the basis of the population graph. They define a graph-theoretic counterpart of the ordinary Voronoi diagram, denoted as discrete weighted Voronoi regions. After applying a heuristic location procedure to define k district centers, the Voronoi regions are determined. The distance between a pair of population units is defined as the length of a shortest path with respect to road distances. Thereafter, an iterative procedure starts incorporating population balance. Distances are updated based on the population of computed regions. This adjustment supports pushing units of (population-wise) heavy districts in directions of light ones. Several variants of the algorithm are executed on randomly generated rectangular grids and instances of Italian regions. The presented computational results are note worthy, especially due to the bad population balance.

Brieden et al. [2017], who presented a paper on constrained clustering, applied their presented approaches on data of parts of German federal states (leaving out larger cities) to achieve districting plans. Their work is based on the close connection between geometric diagrams and clustering. In fact, using the duality of linear programming, the authors work out a relationship between constrained fractional clusterings and additively weighted generalized Voronoi diagrams. First, district centers are heuristically defined, e.g., using the centroids of the current districts in order to obtain similar new districts. A linear program with a population equality constraint is solved with a state-of-the-art solver to achieve fractional assignments of population units to district centers. To come up with integral assignments and to ensure connected districts, some post processing is needed. The centerpiece of this generally described approach is mainly the choice of metrics or more general distance measures. It is worth highlighting that for each cluster, for example, an individual *ellipsoidal* norm can be used. Thus, information regarding current electoral districts can be integrated to achieve a low ratio of voter pairs that used to share a common district but are now assigned to different ones. Depending on the applied metric and post processing, the presented computations need between seconds and several hours to finish.

Every considered publication (except for [Forrest, 1964; Vickrey, 1961]) contains a case study with (real-world) data. Table 2 provides a summary of applications and problem sizes. Additionally, Table 3 offers an overview of the criteria considered. Beyond the criteria mentioned in Table 3, Nagel [1965] and King et al. [2017] also discussed political balance, and Bozkaya et al. [2011, 2003] considered socio-economic homogeneity. A detailed discussion of the implemented measurement functions concerning the requirements of GPDP (cf. Section 3) is provided in Section 5.

4.2 Districting Software

Redistricting software became the predominant tool during the (re)districting process [Altman et al., 2005; Altman and McDonald, 2012]. On the one hand, software is used to analyze current districting plans, organize and evaluate population data, and modify plans manually. On the other hand, driven by the methods and algorithms for the PDP, more and more software provides automated and optimization-based redistricting. A downside is that these is professional software, which is designed to assist decision-makers to perform gerrymandering. In all conscience, we leave out software packages supporting the execution of the malpractice of gerrymandering.

Most of the redistricting software tools are based on a geographic information system (GIS). A GIS allows displaying, managing, analyzing, and capturing

method	citation	application	number of population units	number of electoral districts
exact methods	Garfinkel et al. [1970]	Washington, USA	39	7
	Nygreen [1988]	Wales	38	4
	Li et al. [2007]	New York, USA	62	29
	Kim [2018]	artificial data, grid graph	25, 100	3
col. generation	Mehrotra et al. [1998]	South Carolina, USA	46	6
greedy	Vickrey [1961]	–	–	–
	Bodin [1973]	Arkansas, USA	75	3, 5, 9
location/ allocation	Hess et al. [1963, 1965]	a county in Delaware, USA	?	6
	Hojati [1996]	Saskatoon, Canada	42	11
	George et al. [1993, 1997]	New Zealand	35 000	100
local search	Nagel [1965]	Illinois, USA	90	18
	Kaiser [1966]	Illinois, USA	101	12
	Bozkaya et al. [2003, 2011]	Edmonton, Canada	400	12
	Ricca, Simeone [2008]	5 regions in Italy	246 – 1 208	8 – 28
	Yamada [2009]	Kanagawa Prefecture, Japan	49	17
	King et al. [2017]	4 states in USA	147 565 – 339 933	3 – 29
nature-insp./ probabilistic	Forman and Yue [2003]	3 states in USA	70 – 350	5 – 13
	Bação et al. [2005]	Lisbon, Portugal	52	7
	R. García et al. [2017]	8 states in Mexico	192 – 860	5 – 40
geometric	Forrest [1964]	–	–	–
	Miller [2007]	Washington, USA	1 318	9
	Ricca et al. [2008]	4 regions in Italy	305 – 1 208	8 – 28
	Brieden et al. [2017]	(parts of) 13 states of Germany	52 – 2 304	4 – 48

Table 2. PDP solution approaches in literature and their case study with problem size.

		criteria considered in objective and/or constraints					
method	citation	contiguity	population balance	compactness	adm. boundaries	continuity	other
exact	Garfinkel et al. [1970]	•	•	•			
	Nygreen [1988]	•	•	•	•		
	Li et al. [2007]	•	•	•			
	Kim [2018]	•	•	•			
column gen.	Mehrotra et al. [1998]	•	•	•	•		
greedy	Vickrey [1961]	•	•	•			
	Bodin [1973]	•	•				
location/ allocation	Hess et al. [1963, 1965]	•	•	•			
	Hojati [1996]	•	•	•			
	George et al. [1993, 1997]	•	•	•	•	•	
local search	Nagel [1965]	•	•	•			•
	Kaiser [1966]	•	•	•			
	Bozkaya et al. [2003, 2011]	•	•	•	•	•	•
	Ricca, Simeone [2008]	•	•	•	•		
	Yamada [2009]	•	•				
	King et al. [2017]	•	•	•			•
nature-insp./ probabilistic	Forman and Yue [2003]	•	•	•			
	Bação et al. [2005]	•	•	•			
	R.-García et al. [2017]	•	•	•			
geometric	Forrest [1964]	•	•	•			
	Miller [2007]	•		•			
	Ricca et al. [2008]	•	•	•			
	Brieden et al. [2017]	•	•			•	

Table 3. PDP solution approaches in literature and considered criteria. A • indicates that the criteria of the column is considered in the cited work, either in an objective or (also) as constraints.

characteristics of spatial or geographic data. While editing, e.g., a districting plan, the user perceives the consequences of every change in real time. Altman et al. [2005] reported that in 2001 every US American state (except for Michigan) officially used some kind of redistricting software. Nevertheless, *automated* software was officially employed by very few states [Altman et al., 2005].

In Germany, the Electoral District Commission and its chairman, the Federal Returning Officer, use a software tool called WEGIS (acronym for the German word Wahlkreis-Einteilungs-GIS) [Heidrich-Riske, 2014]. It was developed in-house as a plugin for ArcGIS, a commercial software distributed by the company Esri. WEGIS has been in use since the preparation for federal elections in 2002. In those days, the number of German electoral districts was reduced from 328 to 299. This decision triggered the need for a software tool for supporting redistricting. WEGIS does not provide automated redistricting. It is used for displaying and exporting information, and for facilitating manual redistricting. The software tool is not available to the public. Suggestions for delimiting electoral districts posed by, e.g., political parties, is performed in-house and evaluated by request [Heidrich-Riske and Krause, 2015]. The ArcGIS plugin is specifically tailored to meet German legal requirements. For example, after importing a districting plan and population data, districts exceeding the 15% soft population deviation limit are highlighted in color. This enables the user to quickly spot all districts that should be examined and possibly redrawn.

In the remaining part of this section, we present available software, both commercial and open source, which can be used in the (re)districting process. We distinguish between software that provides an algorithm that can automatically form new districting plans and software enabling only manual modifications. Some of these redistricting tools come with an accompanying scientific publication. More and more tools have become available as web-based applications ensuring that redistricting software is available to millions of non-expert users. However, some software packages are not available to the public, but only to officials or decision-makers of state administrations.

Assisting redistricting by hand

Esri and Caliper are two commercial software vendors that provide licenses for standalone as well as online versions of their redistricting software [Caliper, n.d., online; Esri, n.d., online]. Both systems assist in manual redistricting and are not able to form legal districting plans automatically [Altman and McDonald, 2011, Sec. 6.1]. Owing to their pricing, these programs are not practical for private individuals. Esri and Caliper rather address state and local governments, legislators, and advocacy groups. Several US states used their software in the 1990 and 2000 congressional redistricting [Altman et al., 2005].

Dave's Redistricting App [Bradlee, n.d., online], a free web-based tool, has been developed by an individual software engineer since 2009. Data of every US state (as of 2000 and 2010) is provided and embedded into a mapping service. Furthermore, the population units (voting districts and block groups) can be highlighted in color based on demographic aspects or recent election results. Besides ready-to-use data of US states, own data can be imported. Unfortunately, the tool does not support the common shapefile format. The user can draw electoral districts onto the map and receives population numbers and votes.

Another software package for manual redistricting is DistrictBuilder (not to be confused with the software tool of Bozkaya et al. [2011], which is named exactly the same in their publication). The tool is developed under supervision of authors Altman and McDonald, who have already been cited in the paper. The open-source project allows hosting of online public redistricting initiatives and competitions [Altman and McDonald, 2012]. A software partner builds custom applications as per request.

In order to analyze the 2015 Malaysian districting plan, a non-governmental organization developed a plugin for QGIS, an open-source GIS [Tindak Malaysia, n.d., online]. The free tool comes with population and geographical data of Malaysian states and electoral districts, and enables redistricting by hand, providing several statistics.

Optimization-based redistricting software

AutoBound, distributed by Citygate GIS (formerly known as Digital Engineering Corporation), is a software tool that promises "intelligent automated redistricting" [Citygate GIS, n.d., online]. The product website gives no information about underlying algorithms. According to Altman et al. [2005], a simple greedy multi-kernel growth algorithm as sketched in [Hejazi and Dombrowski, 1996] is used. The vendor states that AutoBound was used for 2000 congressional redistricting in over 30 US states (according to Altman et al. [2005] in only 19 US states) and in Canada for country wide redistricting most recently in 2011. Unfortunately, a demo version of this software is not available.

In a journal paper, Guo and Jin [2011] presented a software called iRedistrict. The system provides optimization-based automated redistricting. Its underlying heuristic is based on a tabu search algorithm, whose performance is evaluated in a study (Iowa, USA: 99 population units, 5 districts). The authors recognize the indispensability of human judgment in presence of criteria that may be vaguely defined and therefore not uniquely quantifiable. The user of iRedistrict can define the weights of the multi-criteria objective, is authorized to select sets of population units to be handled as indivisible units, and can also manipulate computed plans manually. Furthermore, the tool provides useful and customized plots to analyze each objective. iRedistrict can be purchased via the company ZillionInfo as a commercial product [ZillionInfo, n.d., online]. Unfortunately, neither a demo version nor pricing information is available on the website.

A tool called BARD [Altman and McDonald, 2011, online] was presented by Altman and McDonald [2011] in a journal paper. The name is an acronym for "Better Automated Redistricting". BARD is an open-source software package and comes in the form of a module for the R programming language project for statistical computing. The software tool utilizes different procedures for automatically generating plans. The following four metaheuristics are available to refine them: simulated annealing, genetic algorithms, tabu search, and greedy randomized adaptive search [Altman and McDonald, 2011, Section 6.3]. Unfortunately, the software has not been updated since 2011 and is no longer available through the official R module repository.

In Section 4.1, we reviewed the work of Bozkaya et al. [2011, 2003]. Their tabu search heuristic with an adaptive memory procedure was implemented as a plugin for ArcGIS. The authors described how it was used to assist the official designing process of new electoral districts for the city of Edmonton, Canada. One of the authors informed us that their software works fine with ArcGIS version 8 [Bozkaya, 2016]. Unfortunately, this outdated version is not available anymore and the plugin's code has not been upgraded to work with the newest versions of ArcGIS, i.e., as of October 2018, ArcGIS 10.6.1.

The open-source software Auto-Redistrict [Baas, n.d., online] is developed by a private person and includes a genetic algorithm to form districting plans. Details about the genetic algorithm are available on the software's homepage [Baas, n.d., online]. It is possible to load custom shapefiles, to adjust weights of the criteria and to enforce the latter via constraints. It is also possible to shift population units from one district to another by hand. The tool is regularly updated and allows oversight of improvements made by the genetic algorithm in real time as solutions are constantly displayed. Unfortunately, Auto-Redistrict does not support population deviation limits, neither as constraint nor as objective. Just the minimization of squared deviations is possible. Furthermore, one can request equal population as a constraint. Auto-Redistrict supports compactness and some "fairness criteria" concerning bias on the basis of election data [Baas, n.d., online].

In summary, it is unsatisfactory that the majority of presented software tools providing automated redistricting are not available to us for testing. Either, the plugins are outdated and not compatible with current versions of the underlying software, or the districting tools are distributed commercially having no demo version. As presented above, Auto-Redistrict [Baas, n.d., online] is an exception. Including the tools that assist manual redistricting, this software survey highlights that it is a good choice to develop districting software as a plugin of a GIS and to benefit from its features and already implemented functionality. Choosing an open-source GIS, e.g., QGIS, enables any interested person to utilize it.

5 Discussion and Suitability Evaluation for GPDP

To evaluate the suitability of the reviewed PDP solution approaches (cf. Section 4.1) for solving the GPDP, we bring together each publication's considered criteria and the GPDP's constraints as well as objectives. In contrast to Table 3, we will make a careful distinction between whether a criterion is implemented as a constraint or as an objective. Furthermore, we discuss if the concrete measurement of the criteria is rigorous enough for the GPDP. Table 4 contains a summary of the evaluation. The first two columns of Table 4 indicate the superordinate approach as well as the author(s), whereas the remaining columns represent the GPDP's essential criteria (7) – (8) and objectives (9) – (12) (cf. Section 3.3). For each criteria we analyze, if it is considered in the paper's algorithm or model. A "+" indicates that the criterion is implemented in such a way that it could be used without changes for the GPDP. An "o" means that the criterion is taken into account but in a way that is not applicable to the GPDP. No cell entry translates into an omission of the respective criterion. However, this does not imply that it is impossible to adapt the method in this regard.

In the following, we discuss our findings in detail. We take a closer look at literature's measurements of the GPDP's objectives (9) and (10), i.e., continuity and administrative conformity, since we did not cast them in mathematical terms in Section 3.3 and this does not seem to be trivial either.

method	citation	constraints (7) and (8)		objectives (9) – (12)			
		contiguity	max pop. deviation	continuity	adm. boundaries	tol. pop. deviation	pop. balance
exact	Garfinkel et al. [1970]	+	+				+
	Nygreen [1988]	+	+		o		+
	Li et al. [2007]	o	+				o
	Kim [2018]	+					+
column gen.	Mehrotra et al. [1998]	+	+		o		o
greedy	Vickrey [1961]	o	+				o
	Bodin [1973]	+					+
location/ allocation	Hess et al. [1963, 1965]	o	+				
	Hojati [1996]	o	+				
	George et al. [1993,1997]	o	+	+	o	+	+
local search	Nagel [1965]	+	+				
	Kaiser [1966]	o	+				+
	Bozkaya et al. [2003,2011]	+	o	+	o	+	+
	Ricca, Simeone [2008]	+			o		+
	Yamada [2009]	+					+
	King et al. [2017]	+	+				+
nature-insp./ probabilistic	Forman and Yue [2003]	o	o			+	
	Bação et al. [2005]	o	o				+
	R.-García et al. [2017]	o	o				+
geometric	Forrest [1964]	o	o				o
	Miller [2007]	+					
	Ricca et al. [2008]	+					o
	Brieden et al. [2017]	+		o			+

Table 4. Overview of the criteria considered in the literature.
+: Criterion considered and implemented according to the formulation of the GPDP.
o: Criterion considered, but not rigorously enough for the GPDP.

The size of the GPDP instances are by far greater than the instances solved by exact methods in the literature (cf. Table 1 and 2). Since these results are up to almost 50 years old, one should investigate if and to what extent today's solvers and computer technologies can handle larger instances using these models. There is no doubt that the exact method of Garfinkel and Nemhauser [1970] becomes more promising through the reasonable embedding of Mehrotra et al. [1998] in a branch and price approach. Mehrotra et al. [1998] apply a postprocessing step in which population between districts is shifted in line with the objective of minimizing the number of split population units, in this case counties. Overall, this can be seen as an weak implementation of administrative conformity which is clearly not rigorous enough for the GPDP.

Nygreen [1988] considers conformity to administrative boundaries insofar as the author forces all population units of the same city to belong to the same electoral district. This implementation is insufficient for the GPDP since the criterion of administrative conformity is far more comprehensive in the German case. The model of Li et al. [2007] is not compatible with the definition of the GPDP either. For example, there is no guarantee that this formulation will produce contiguous districts, although this is favored in the objective function. From a practical perspective, their assumption to split population units at any position is debatable. This requires additional effort to transform a solution into a legal districting plan.

The contiguity model of Williams [2002a,b] used by Kim [2018] should be pursued further, of course following the note by Validi and Buchanan [2018]. The formulation could also be implemented in a pricing problem as in [Mehrotra et al., 1998] to ensure contiguity of generated electoral districts. This would remove the disadvantage of the model of Mehrotra et al. [1998], since Williams' formulation encompasses all connected subgraphs while Mehrotra et al. [1998] ignores some. An exact solution method based on branch and price would be the outcome. As stated, Williams [2002b] utilizes planarity of the used graph. For the GPDP, the population graph is not always planar. There are municipal areas that do not themselves form a contiguous area, and this results in a population graph not being planar [Goderbauer, 2016, Example 8.3]. However, one can imagine some preprocessing to obtain planarity in the GPDP instances.

All considered multi-kernel growth methods are not suitable for the German case due to the wide diversity of criteria and objectives considered in the GPDP. It seems to be inappropriate to incorporate more criteria than contiguity and population balance in such greedy algorithms. A greedy setting seems to be unqualified especially for considering conformity to hierarchically structured administrative boundaries. However, since such algorithms are very fast, they may be used to compute a starting solution. For example, this is the case in [Bozkaya et al., 2011, 2003].

Location-allocation approaches mentioned in the literature on the PDP have a simple but fundamental drawback: They do not ensure contiguity of resulting electoral districts. Nevertheless, the location-allocation method of George et al. [1993, 1997] managed to consider more or less all criteria and objectives of the GPDP. As mentioned before, George et al. solves a minimum-cost network flow problem in the allocation step. Using different arc costs in the underlying network, almost every imaginable objective can be modeled. To give an example, George et al. penalizes each crossing of natural barriers (e.g., mountain ranges, rivers) with a constant. However, this does not encompass the multilevel GPDP objective of conformity to administrative boundaries. To support continuity in the districting plan, it is penalized if a population unit is assigned to a district different from a previously given districting plan. This penalty is implemented as arc costs of the mentioned network and depends on the distance between population-wise centers of gravity of units and districts. It should be noted that George et al. provides different versions of their model, each incorporating a subset of all discussed objectives. On the one hand, this illustrates the flexibility of their approach. On the other hand, they bypass the difficulties of the multi objective nature of the problem and the trade-off between the different objectives.

Considering the local search algorithms in Table 4, the work of Bozkaya et al. [2011, 2003] stands out from others. Their tabu search algorithm considers most of the essential criteria and objectives of the GPDP. Contiguity is treated as the only hard constraint. All other criteria are implemented through measures combined into a weighted additive multicriteria function. According to the authors, they propose a new measure in order to compare similarity of a computed districting plan with an existing plan. Their continuity index endorses districts which have large overlapping areas with an existing district. This index can be used even if old and new plans do not contain the same number of districts. However, since it considers the overlapping area of regions, this measure serves more the visual continuity between districting plans – which certainly can be a legitimate objective. Owing to the vast differences in population density and the interpretation that the goal of continuity refers to population (as the most important component in an democratic election), this measure is potentially debatable at least for the GPDP. In contrast, measuring the district overlay by means of involved population may be a small but suitable modification of the similarity index proposed by Bozkaya et al. [2011, 2003].

In addition, Bozkaya et al. [2011, 2003] implemented a criterion called *integrity of communities*, which requires that communities with common interests be kept within the same electoral district. In the context of electoral districts in the Canadian city of Edmonton, Bozkaya et al. [2011] give the example of French-speaking communities. From the point of view of administrative units as communities of interest, it would be interesting to rephrase this criterion as the GPDP's administrative conformity and analyze if it is appropriate. Bozkaya et al. [2011, 2003] defines $f_{\text{int}}(\mathcal{D})$ as the measurement of integrity of communities for a districting plan $\mathcal{D} = \{D_1, \ldots, D_k\}$. As an objective function which is to be *minimized* the measurement reads

$$f_{\text{int}}(\mathcal{D}) := 1 - \frac{\sum_{\ell=1}^{k} G(D_\ell)}{\sum_{i \in V} p_i}$$

where $G(D_\ell)$ represents the population of the most represented community in electoral district D_ℓ. As before, $\sum_{i \in V} p_i$ equals the total population of the PDP instance.

In terms of the GPDP, we consider every rural and urban district as a community of interest. As per the definition of f_{int}, electoral districts D_ℓ with $G(D_\ell) = \sum_{i \in D_\ell} p_i$ contribute in the best possible way to the objective function. These electoral districts contain only units of *one* community of interest, i.e., rural or urban district, regardless of whether the electoral district coincides exactly with the administrative unit or comprises only a part of it. That is suitable for the GPDP. For example, with regard to urban/rural districts, this is the case for (i) the electoral district which matches exactly with the rural district of Warendorf in North Rhine-Westphalia and (ii) each electoral district of the city and urban district of Munich in Bavaria [Federal Returning Officer, n.d., online]. However, there are German electoral districts which are identical to up to four urban and rural districts. In German practice, this is as good to evaluate as an electoral district which is exactly identical to *one* administrative area. Unfortunately, this fact is not taken into account and actually penalized in the conformity index by Bozkaya et al. [2011, 2003].

Ricca and Simeone [2008] used an administrative conformity index which is not described in detail in their publication but in [di Cortona et al., 1999, Section 11.3]. We review the proposed administrative conformity index in detail and explain why it is not suitable for the GPDP.

Let h be a type of administrative area, e.g., h indicates the level of rural/urban districts. Let A_h denote the number of administrative areas of type h. The conformity index $C(D_\ell, h)$ for electoral district D_ℓ and administrative area type h is based on the distribution of district's units $i \in D_\ell$ among the areas of type h: Let $\delta_{\ell,a}$ denote the number of units $i \in D_\ell$, which belong to area $a \in \{1, \ldots, A_h\}$ of administrative area type h. di Cortona et al. [1999] define the conformity index which has to be *maximized* as

$$C(D_\ell, h) := \frac{1}{|D_\ell|^2} \sum_{a=1}^{A_h} \delta_{\ell,a}^2.$$

The index $C(D_\ell, h) \in [\frac{1}{A_h}, 1]$ is maximal, i.e., $C(D_\ell, h) = 1$, if D_ℓ contains only units of *one* administrative area of type h. The proposed conformity index is minimal, i.e., $C(D_\ell, h) = \frac{1}{A_h}$, when units $i \in D_\ell$ are equally distributed among all A_h administrative areas of type h. A global conformity index is defined as average over all districts and all types of administrative areas.

In the German context, the weakness of the conformity index proposed by di Cortona et al. [1999] is the same as pointed out for the work of Bozkaya et al. [2011, 2003]: The measurement penalizes if an electoral district exactly matches more than one administrative area of one type. Thus, the proposed measurement of di Cortona et al. [1999] is not suitable for an administrative level containing numerous areas which are too sparsely populated to define their own electoral district.

Using the framework of nature-inspired and probabilistic algorithms, Forman and Yue [2003], Bação et al. [2005], and Rincón-García et al. [2017] consider the districts' contiguity only in a single fitness function or in an additional contiguity check. Neither continuity nor administrative conformity is regarded. Of course, this fact does not exclude the concept of these metaheuristics for being adequate to solve the GPDP but rather leaves room for further research.

Algorithms using Voronoi regions by Miller [2007] and Ricca et al. [2008] focus mainly on maximizing the compactness of the electoral districts. In contrast to the PDP in the USA, for example, compactness is not a (primary) goal to achieve in the German case. It is widespread in the PDP literature and especially in Voronoi approaches to use squared Euclidean distances or road distances to achieve compactness. In this respect, the work of Brieden et al. [2017] is refreshing. The authors apply an individual ellipsoidal norm for each electoral district in their anisotropic power diagram approach. Since these norms are computed on the basis of pre-given electoral districts, it favors the computation of similar districts. Nevertheless, this approach strives for continuity only implicitly. Brieden et al. [2017] evaluate the extent of continuity after the solution computation, namely by the ratio of voter pairs that are used to share a common district but are assigned to different ones in the solution output.

By summing up the suitability evaluation of the solution approaches for GPDP, we come up with the following three aspects.

Firstly, a column generation/branch and price approach as proposed by Mehrotra et al. [1998] seems promising. Besides the previous related work of Garfinkel and Nemhauser [1970], the implicit enumeration of Mehrotra et al. [1998] is one of the few that ensures the two essential criteria of the GPDP (see Table 4). Mehrotra et al. [1998] identifies the compactness of each generated district with its costs in the objective. The sum of costs is minimized in the set partitioning problem. It is possible to consider more diverse costs and thus to make the approach suitable for the GPDP. As mentioned, only subtrees of shortest path trees are considered as possible electoral districts in the pricing problem. Therefore, Mehrotra et al. [1998] provide only an optimization-based heuristic. Using another model to ensure a connected subgraph in the pricing problem, e.g., [Williams, 2002b] with [Validi and Buchanan, 2018], can eliminate this drawback, since the technique of branch and price can solve problems exactly.

Second, the local search heuristic of Bozkaya et al. [2011, 2003] nearly fits each requirement of the GPDP. It is possible to consider more diverse and GPDP-tuned costs. Concerning the measurement of continuity, this can easily be done by using the population number as a basis for assessment rather than the surface area. The speed-up of continuity checks as provided by King et al. [2017] should be implemented in a local search.

Third, the PDP literature does not provide any measurement for conformity of administrative boundaries completely fulfilling all requirements of the objective of the GPDP. As described before, every suggestion has drawbacks regarding the hierarchical multi-level character of administrative divisions in Germany. Moreover, that electoral districts which are part of exactly one rural/urban district and electoral districts which exactly match a number of rural/urban districts should be rated well.

6 Summary and Outlook

In this work, we examine the optimization problem of partitioning a territory into electoral districts: the Political Districting Problem (PDP). We provide a unified and extendable formulation of the PDP, based on two basic criteria: contiguity and population balance. As has been pointed out, this leads to an NP-hard problem. As a specific application, we consider the German Political Districting Problem (GPDP). We introduce the German electoral system and point out the significance and topicality of the GPDP in ongoing (political) discussions. We present all legal requirements for German electoral districts and define the GPDP as a multi-objective partitioning problem. The PDP is widely discussed in the literature. We review solution approaches, models, and algorithms proposed for the PDP. Only a few published solution approaches solve the PDP exact, the focus is clearly on heuristics. Various software packages are offered which provide assistance for state administrations during the redistricting process, or enable interested citizens to analyze and compute districting plans. Unfortunately, most software is only commercially available and some open-source projects are outdated.

The review of the exact solution approaches illustrates that reported computational results are approximately as old as 50 years. One should investigate to what extent todays technologies can handle larger instances. Ensuring contiguity efficiently seems to be an issue in exact methods. Furthermore, in most cases, the solution methods provided in the literature are green-field approaches that do not utilize current districting plans. In practice, however, a districting plan is often given and has to be adjusted, preferably as little as possible. One can focus on combinatorial redistricting problems occurring in connection with the regular adjustment of districting plans, thereby combining complexity questions of the occurring (continuity) problems with application-oriented answers for decision-makers.

Our literature review reveals that the German case differs from the most widely discussed PDP variants in the following aspects. Continuity is rarely considered in the literature. In Germany, however, it is a very important objective and, in general, a quite natural one. The number of electoral districts *with respect* *to German federal states* changes sometimes. Consequently, attention should be paid to the objective of continuity also in case of increasing or decreasing the number of electoral districts. In most approaches in the literature, compactness is a fundamental objective. In Germany, neither legal requirements, nor court decisions nor exterior discussions call for (maximally) compact electoral districts. In Germany, it is important to favor conformity between electoral districts and administrative boundaries. This includes several levels of the hierarchical administrative structure. In a sense, pursuing this conformity implicitly leads to compact electoral districts. As we conclude from the literature review, no suitable measurement for this objective has been proposed to date. Having one population deviation limit as a constraint (maximum limit of 25%) and another within an objective (tolerance limit of 15%) makes the GPDP unique. The GPDP consists of subproblems, in which sizes (measured by the size of population graph on municipality level) surpass most test instances studied in the literature.

On the whole, the GPDP stands out from classical PDPs in various aspects. Therefore, we think that studying the GPDP with its associated constraints and objectives in detail would enrich the literature on the PDP.

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