Subproblem Solving in Benders Decomposition for Affine Potential-Based Flow Problems with Topology Switching and Robustness Scenarios

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Potential-based flow networks [3,10,12] are used to model problems where node potentials $\pi_u \in \mathbb{R}$ and arc flows $x_{(u,v)} \in \mathbb{R}$ satisfy the potential equation $\pi_u - \pi_v = \mu_{(u,v)}^{-1}\psi_{(u,v)}(x_{(u,v)})$, with arc conductance $\mu_{(u,v)} \in \mathbb{R}$ and potential function $\psi_{(u,v)} : \mathbb{R} \to \mathbb{R}$. Examples are electrical circuits [4], gas pipelines [13], traffic assignment [8], and other applications [12]. Changes in topology can be modeled by switch arcs, making the problem NP-hard in general [9]. Switching has been combined with robustness against the outage of one or more edges [6,7]. Such problems are decomposable via Benders decomposition [2], where the master problem (MP) decides topology and then subproblems (SPs) evaluate the scenarios. Based on SP dual solutions, optimality or feasibility cuts are generated and added to the MP, guiding towards a globally optimal solution. Standard implementations solve SPs as linear programs, e.g., using simplex [11].

Contribution. We reduce problems with affine potential functions $\psi_{(u,v)}(x_{(u,v)}) = r_{(u,v)}x_{(u,v)} + s_{(u,v)}$, $r_{(u,v)} > 0$, $s_{(u,v)} \in \mathbb{R}$, used primarily in lossless DC power flow problems, to equivalent problems with $\psi_{(u,v)}(x_{(u,v)}) = x_{(u,v)}$. We present a fast method to obtain Benders optimality and feasibility cuts: SP primal solutions can be uniquely determined by a system of linear equations (SLE), solved using LU-factorization. Using the same factorization, we find an optimal dual solution. We exploit the similarity between scenarios by reusing factorizations via low-rank updates. The Benders iteration complexity for single-edge outages is cubic in the number of nodes. We demonstrate the method on security-constrained optimal transmission switching (SCOTS) instances, where runtime reductions of up to 98.80 % are observed compared to a standard LP-based approach.

Problem Statement. We consider graphs $(V, A_L \cup A_S)$ with nodes V, line arcs $A_L \subseteq V \times V$, disjoint from switch arcs $A_S \subseteq V \times V$. Each node $u \in V$ has potential $\pi_u \in \mathbb{R}$ and demand $b_u \in \mathbb{R}$ with $\sum_{u\in V} b_u = 0$. Its in- and outgoing arcs are denoted by $\delta^-(u), \delta^+(u) \subseteq A_L \cup A_S$. Each arc $(u,v) \in A_L$ has flow $x_{(u,v)} \in \mathbb{R}$, conductance $\mu_{(u,v)} > 0$, and satisfies the potential equation $\pi_u - \pi_v = \mu_{(u,v)}^{-1}\psi_{(u,v)}(x_{(u,v)})$, where $\psi_{(u,v)} = r_{(u,v)} \cdot x + s_{(u,v)}$, $r_{(u,v)} > 0$, $s_{(u,v)} \in \mathbb{R}$, is its potential function. Flows are bounded by $X_{(u,v)}^{\min}, X_{(u,v)}^{\max} \in \mathbb{R}$. They incur costs $c^> \in \mathbb{R}$ per unit surpassing the soft bounds $X_{(u,v)}^{>,\min}, X_{(u,v)}^{>,\max} \in \mathbb{R}$. Switch arcs $(u,v) \in A_S$ have binary state $z_{(u,v)} \in \{0,1\}$. When on ($z_{(u,v)} = 1$), potentials are equal ($\pi_u = \pi_v$). Conservation for flows and demands holds at every node.

Reduction to Identity Function. For each line arc $(u,v) \in A_L$, we replace the potential function with the identity function $\mathrm{id}(x) = x$. We modify conductances, bounds and demands (denoted by superscript "id") to obtain an equivalent problem. It holds that

$$x_{(u,v)} = \psi^{-1}_{(u,v)}\left((\pi_u - \pi_v)\cdot \mu_{(u,v)}\right) = \left(\frac{(\pi_u - \pi_v)\cdot \mu_{(u,v)} - s_{(u,v)}}{r_{(u,v)}}\right)$$
$$= \mathrm{id}^{-1}\left((\pi_u - \pi_v)\frac{\mu_{(u,v)}}{r_{(u,v)}}\right) - \frac{s_{(u,v)}}{r_{(u,v)}}.$$

We set the modified conductance $\mu^{\mathrm{id}}_{(u,v)} := \mu_{(u,v)}/r_{(u,v)}$. The remaining constant part $-s_{(u,v)}/r_{(u,v)}$ is added to the flow bounds, e.g., $X^{\mathrm{id},\min}_{(u,v)} = X^{\min}_{(u,v)} - s_{(u,v)}/r_{(u,v)}$. Additionally, we adjust the demands by

$$b^{\mathrm{id}}_u = \sum_{(u',v')\in\delta^-(u)} \frac{s_{(u',v')}}{r_{(u',v')}} - \sum_{(u',v')\in\delta^+(u)} \frac{s_{(u',v')}}{r_{(u',v')}} + b_u \quad \forall u \in V,$$

Benders Loop. We solve one MP and then multiple SPs. We omit the original and MP formulations here for brevity. The MP fixes switch arcs $\overline{z}_{(u,v)} \in \{0,1\}$, $(u,v) \in A_S$, and demands $\overline{b}_u \in \mathbb{R}$, $u \in V$. For each outage scenario ("contingency") $c \in \mathcal{C}$, the SP is:

$$\begin{aligned}
\min \quad & \sum_{(u,v)\in A^c_L} c^{>} x^{>,c}_{(u,v)} \\
\text{s.t.} \quad & \sum_{(v',u)\in\delta^-(u)\setminus c} x^c_{(v',u)} - \sum_{(u,v')\in\delta^+(u)\setminus c} x^c_{(u,v')} = \overline{b}_u \ [\varphi^c]\ \forall u \in V && (1)\\
& (\pi^c_u - \pi^c_v)\mu_{(u,v)} = x^c_{(u,v)} \quad [\omega^c]\ \forall (u,v) \in A^c_L && (2)\\
& X^{>,\min}_{(u,v)} - x^{>,c}_{(u,v)} \le x^c_{(u,v)} \le X^{>,\max}_{(u,v)} + x^{>,c}_{(u,v)} \quad [\sigma^c]\ \forall (u,v) \in A^c_L && (3)\\
& X^{\min}_{(u,v)} \le x^c_{(u,v)} \le X^{\max}_{(u,v)} \quad [\tau^c]\ \forall (u,v) \in A^c_L && (4)\\
& (1 - \overline{z}_{(u,v)})(\pi^{\min}_u - \pi^{\max}_v) \le \pi^c_u - \pi^c_v \\
& \qquad \le (1 - \overline{z}_{(u,v)})(\pi^{\max}_u - \pi^{\min}_v)\ [\alpha^c]\ \forall (u,v) \in A_S && (5)\\
& \pi^c_u \in \mathbb{R} \quad \forall u \in V\\
& x^c_{(u,v)} \ge 0 \quad \forall (u,v) \in A_S\\
& x^c_{(u,v)} \ge 0 \quad \forall (u,v) \in A^c_L\\
& x^{>,c}_{(u,v)} \ge 0 \quad \forall (u,v) \in A^c_L
\end{aligned}$$

It determines potentials π^c_u, $u \in V$, flows $x^c_{(u,v)}$, $(u,v) \in A^c_L$, and overload $x^{>,c}_{(u,v)}$, $(u,v) \in A^c_L$. We assume that the active graph in each subproblem is connected, this is ensured by a separate separation algorithm.

Primal Solution. We contract switched-on switch arcs by merging incident nodes. This reduces the primal subproblem to determining node potentials satisfying flow conservation:

$$\sum_{(v',u)\in\delta^-(u)\setminus c} \mu_{(v',u)}(\pi^c_{v'} - \pi^c_u) - \sum_{(u,v')\in\delta^+(u)\setminus c} \mu_{(u,v')}(\pi^c_u - \pi^c_{v'}) = \bar{b}_u \quad \forall u \in V,$$

$$\pi^c_{|V|} = \pi^{\mathrm{ref}},$$

where $\pi^{\mathrm{ref}} \in \mathbb{R}$ is an arbitrarily chosen reference potential to ensure solution uniqueness. We denote this SLE by $D_p\pi^c = b_p$, where $\pi^c \in \mathbb{R}^{|V|}$ is the vector of potentials. It has a unique solution [10]: let $D'_p \in \mathbb{R}^{(|V|-1)\times(|V|-1)}$ be the reduced matrix after fixing the reference potential $\pi^c_{|V|}$ to π^{ref}. Its entries are

$$D'_{p,(u,v)} = \begin{cases} \sum_{(u',v')\in\delta(u)\setminus c} \mu_{(u',v')}, & u = v, \\ -\mu_{(u,v)}, & u \neq v \text{ and } (u,v) \in A^c_L, \\ -\mu_{(v,u)}, & u \neq v \text{ and } (v,u) \in A^c_L, \\ 0, & \text{otherwise.} \end{cases}$$

This matrix is (i) weakly diagonally dominant with positive diagonal entries, (ii) strictly dominant for at least one row (graph connected), and (iii) irreducible. This implies full rank and, by Rouché-Capelli, that a unique solution exists.

Optimality Cuts. To construct Benders optimality cuts, we require dual variables of an optimal dual solution. Again, we at first contract switch arcs. Given an optimal primal solution $(\pi^{c*}, x^{c*}, x^{>c*})$, the dual variables $\sigma^{c*}_{(u,v)}$ for Constraints (3) of an optimal dual solution satisfy by complementary slackness

$$\sigma^{c*}_{(u,v)} = \sigma^{\min,c*}_{(u,v)} - \sigma^{\max,c*}_{(u,v)} = \begin{cases} 0, & X^{>,\min}_{(u,v)} \leq x^{c*}_{(u,v)} \leq X^{>,\max}_{(u,v)}, \\ -c^>, & x^{c*}_{(u,v)} < X^{>,\min}_{(u,v)}, \\ c^>, & x^{c*}_{(u,v)} > X^{>,\max}_{(u,v)}, \end{cases} \quad \forall (u,v) \in A^c_L.$$

The dual variables φ^{c*}_u are then determined with the SLE $D_p\varphi^c = D_d\varphi^c = b_d$

$$\sum_{(v',u)\in\delta^-(u)\setminus c} \mu_{(v',u)}(\varphi^c_u - \varphi^c_{v'}) - \sum_{(u,v')\in\delta^+(u)\setminus c} \mu_{(u,v')}(\varphi^c_{v'} - \varphi^c_u)$$

$$= \sum_{(u,v')\in\delta^+(u)\setminus c} \sigma^{c*}_{(u,v')} - \sum_{(v',u)\in\delta^-(u)\setminus c} \sigma^{c*}_{(v',u)} \quad \forall u \in V,$$

$$\varphi^c_{|V|} = \pi^{\mathrm{ref}}.$$

Following that, the potential equation dual variables are computed as

$$\omega^c_{(u,v)} = \varphi^{c*}_u - \varphi^{c*}_v - \sigma^{c*}_{(u,v)} \quad \forall (u,v) \in A^c_L.$$

We observe that $D_p = D_d$, so the same arguments for uniqueness hold. In particular, the reference value $\pi^{\mathrm{ref}} \in \mathbb{R}$ can be chosen freely: the dual objective contains the term $\sum_{u\in V} \varphi^c_u \bar{b}_u$, but $\sum_{u\in V} \bar{b}_u = 0$, so $\sum_{u\in V}(\varphi^c_u + \varepsilon)\bar{b}_u = \sum_{u\in V} \varphi^c_u \bar{b}_u$ for any $\varepsilon \in \mathbb{R}$. Switch arc duals τ^{c*} are computed via spanning tree propagation for each component of the graph restricted to active switch arcs.

Feasiblity Cuts. A feasibility cut is added if the hard bounds of an arc $(u, v) \in A_L$ are violated. We modify the SLE for optimality cuts. The coefficient matrix has an additional column with entries 1 and -1 for u and v. This accounts for the arc's dual variable for Constraint (5). An additional row sets the dual objective equal to a positive value on the right-hand side. The right-hand side is otherwise equal to 0. A solution of this SLE yields a dual direction of unboundedness.

Factorization Updates. Optimality and feasibility cuts share similar matrix structures. Between subproblems, only outage arcs change, each handled by two rank-1 updates to LU-factorizations. For the complete set of singleton outages, complexity reduces from $\mathcal{O}(|V|^4)$ to $\mathcal{O}(|V|^3)$ per Benders iteration when reusing factorizations. Updates can also be used between Benders iterations.

Computational Results. We use instances from pglib-opf [1] (up to 1888 nodes, 2531 scenarios) and Gurobi as solver [5]. Preliminary results show that using SLEs over LP solvers reduces mean total (SP) solving time by ca. 78.20 % (96.50 %), 84.30 % (98.80 %) with factorization updates.

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